【附件三】**教育部教學實踐研究計畫成果報告格式(系統端上傳 PDF 檔)**

教育部教學實踐研究計畫成果報告(封面)
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教學實踐研究計畫成果報告

關於數學系高等微積分教學的新嘗試

(1) 研究動機與目的

本人相信未來的教育必將走向個人化，走向每一個人。考慮到台灣少子化問題，“一個都不能少” (No-One-Left-Behind Policy) 顯得更為重要。所以本教學計劃的目標就是要開始一種 “一對一的大學教育” 模式的嘗試。

本人試圖解決之問題是，用最狹義的方式（或教學現場）說明，是“提高數學系高等微積分之理解程度與及格率”。當然在大量不及格的背後隱藏著一連串的問題。問題的嚴重性可以參考這門課的如下四個特點：

1. 其不及格率遙遙領先數學系其他任何一門課。全班大部分人都被擋是見怪不怪。說到數學系有一門課讓大家談虎色變，大家馬上會想到高微。
2. 數學系大部分延畢都與之相關。因為是必修，所以導致大量重修，擋修。
3. 每年數學系排助教時，第一優先一般是先定最好的學生做高微的助教。然後再排其他課。雖然它只是大二的課程，但數學系約六十名碩士生中很難找到能勝任助教的工作。因為他們大多不懂。

高微是以往研究所碩士班招生的主要科目，因為是數學系大學部大部分進階課程的基礎。但兩年前開始，因為擔心要求考高微會嚇阻學生報名而取消其考試。

本人試圖解決之問題，用廣義的方式則是提高學生深度思考的能力。這一點尤其在現在的所謂數位時代，學生習慣大量信息的強刺激，所以在信息量提升的同時，他/她們的 attention span 與深度思考能力似乎在下降。

(2) 文獻探討

我們一對一的方式在教學中體現在三個方面：

1. Individual help, to help students get started;
2. Instant feedback, to help students grow strong;
3. Interactive textbook, to help students dig deeper in thinking.

我們下面針對這幾方面列舉一些文獻。

首先近幾年台灣最引人注目的翻轉教學(flipped classroom)取得了巨大的成功。關於微積分，主要體現在大量的 online courses 即遊戲化的微積分，同時也有提供一些及時發問的機會。最有名的大概是台大的“微積分電競大賽”與“微積分之夜”。

1. https://www.ntu.edu.tw/spotlight/2016/780_20160408.html
2. https://m.facebook.com/events/1137603106349582/

很明顯這是針對非高等微積分。解決的問題是計算與技巧，而非深入思考的提高。事實上我們在本計劃第六頁有提到電競化如果用於高等微積分有益否其實是值得思考的。

國外也有類似遊戲化軟體，如開發 Variant 微積分遊戲的 Triseum 公司：

3. https://triseum.com/calculus/variant/

關與網上的課程，不論微積分或高等微積分，不論中文或英文，均有大量。茲舉數例：

4. http://ocw.aca.ntu.edu.tw/ntu-ocw/index.php/ocw/cou/103S121
5. http://ocw.nthu.edu.tw/ocw/index.php?page=mobile&type=course&cid=204
6. http://ocw.nctu.edu.tw/course_detail.php?bgid=1&gid=1&nid=9

它們的優點顯而易見，如同大量其他科目一樣。但對我們關心的高微（特有）的兩大問題：(a) 聽了也不懂；(b)懂了也不能自己動手做，似乎幫助不大。

下面我們討論及時回饋（instant feedback）.這方面的軟體近幾年出現了不少。如最近在台灣各大學盡力推廣的 Zuvio。

7. https://www.zuvio.com.tw/

本人對之饒有興趣。特別參加過其專門的 workshop，但發現它與本人之前了解的及時回饋軟體有類似的問題：即它對處理文字，圖片，甚至影音，都甚方便，但無法處理（稍微複雜一點的數學符號），對偏重邏輯推理的內容也不太方便。而且本人有詢問開發者，似乎這個問題短期內無法解決。其它現有的類似軟體如 Socrative 或 habook：

8. https://www.socrative.com/
9. https://www.youtube.com/watch?v=36eV6F2DXoA&feature=youtu.be
10. http://www.habook.com.tw/eTeaching/
11. https://www.youtube.com/watch?v=CB3i8I-UdV8&feature=youtu.be

其實本人對使用類似“智慧教室”系統，非常感興趣，但又對它們該如何用於偏重深入思考的科目則相當謹慎。

下面我們討論一對一的課程或幫助。在台灣，一對一主要出現在三種場合：(a) 補習班，(b)英文（外語）學習 (c)手作類，如木工。從教育理念上探討的則甚少

12. http://jbn.education/individual-self-exploration/

上面的“家百濃教育諮詢”似乎是少有的類似機構。其偏重大學前的教育。

在國外最有名的，類似教育思路大概是牛津大學與劍橋大學的導師制。以人口僅六千四百萬而言，英國培育出大量學術精英，與其導師制不無相關。這方面文獻甚多，玆舉兩網頁以便快速瀏覽：

13. https://www.undergraduate.study.cam.ac.uk/courses/how-will-i-be-taught
14. https://www.theguardian.com/education/2011/jan/25/cambridge-may-end-individual-tuition

至於美國，其實類似功能由其相當龐大的 REU program（Research Experience for Undergraduates）所代替。本人在美國任教的十多年中，多次參與，可以說美國本國人中成長為學術人才的很少有人沒

參與過。其他大量沒走學術道路的通過 REU 也對研究能力得到課堂上無法獲取的訓練。

15. https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5517
16. http://www.ams.org/programs/students/emp-reu

在台灣類似 REU 的功能 NCTS 與中研院有提供，但規模比較小。本人認為，未來的教育，以台灣人口減少及未來無自然資源可依靠而言，應該將參與類似 REU 的機會的規模擴大到至少一半的大學生。

至於以一對一為根本辦學理念的學校，一般為（極其）昂貴的私立學校。如

17. http://www.halstromacademy.org/one-to-one.html
18. https://www.sarahlawrence.edu/about/

前者為中小學教育，其網頁提供參考的年學費為 $29400. 後者為美國著名文理學院，年學費約 $65480 （參考如下。）.

19. https://www.infoplease.com/us/higher-education/most-expensive-us-colleges-and-universities-2015-2016

台幣幾百萬的學費在台灣大概無法複製，但其部分理念應該是未來教育趨勢。尤其是後者被 "named the higher education institution with the "best classroom experience" in all of America by Princeton Review in 2016." （參考如下）

20. https://www.omicsonline.org/universities/Sarah_Lawrence_College/

最後關於一對一教學在大學層面的理論探討甚少，事實上在如下一文中指出： "The one-to-one movement is still young and careful long-term studies are hard to find. "

21. Bruneau, O.J. (2015). One-to-one technology: Meeting expectations? Oneota Reading Journal.

音樂領域也許是一對一少有成為普遍教學方式的。所以文獻較多，如

22. Gaunt， H. （2011） Understanding the One-to-One Relationship in Instrumental/Vocal Tuition in Higher Education: Comparing Student and Teacher Perceptions，British Journal of Music Education, volume 28, 159-179.

其他一對一相關文獻則較多是指每人配一台電腦或其它設備，如：

23.Dunleavy, M., Dexter, S., & Heinecke S. (2007). What added value does a 1:1 student to laptop ratio bring to technology-supported teaching and learning? Journal of Computer Assisted Learning 23, 440 – 452.
最後，關於互動式，個人化的電子教材，是本計劃的重點之一，但似乎之前沒有任何人有任何嘗試，所以特此註明，沒有找到相關文獻。但本人在過去四年來中大後帶碩士生/博士生時，已積累幾千頁的經驗，似乎對學生效果很好。

(3) 研究方法

本次研究的方法更多是在實務操作上，與學生增加一對一交流的機會，希望能提高他們的思考能力，與人互動交流的能力，當然也提高學業成績。

我們一共有十二名助教參與：陳元浩，徐子翔，朱文滔，何宜融，洪嘉陽，陳文威，陳姿穎，黎懷仁，黃聖傑，郭立峰，卓冠宇，劉超（交換博士生，未支薪）。每週三晚上從 7pm 開始，大約持續兩到三個小時，提供大量個別輔導的機會。

本人很希望能引入統計學中量化方式來做定量分析，也試著尋找有這方面能力的助理。但是發現不太容易。本校教學中心的回饋意見是，技術人力是最貴也最難得的。只能靠計劃執行者自己想辦法。想必有不少同仁有類似困惑。所以如果教育部能發展一些線上工具，應該很有幫助。

(4) 教學暨研究成果

主要可複製並反复使用的成果有兩項：

一是電子式，互動教材，含 latex 一份及對應 pdf 文檔（85 頁，見附件 A&B）。

這可以讓班上每一位同學自由複製，然後個人使用當中自由調整和修改，加入自己的理解和問題。

也可以讓以後的類似課程直接複製。

二是自編上課留白式講義，約三百頁，見附件 C&D。另外有比較多零星講義則未附上。

這些講義中留白的目的是提高上課效率，促使學生不是忙於記筆記，而是當場能夠思考和互動。

另外同上面的回饋，本人很希望能引入統計學中量化方式來做定量分析。但限於技術能力，只好作罷。但本學年的通過率與理解能力應該確實有大幅度提高。從以往約一半被當掉到大約十幾趴，當然這裡具體數字也許不是太有意義，因為可能說與老師拿捏程度有關。但我們在計劃中的標準是每一個學生有任何問題我們都有人力提供及時的回答，並且我們基本上全程確切掌握每一個學生的狀況。這樣我們可以說每一個學生，如果最終有意外，不是因為課堂上的原因。而是一些個人因素。

因為我們投入了幾乎龐大的人力，所以對每一個學生的狀況都比較了解，我們幾乎全程跟進和幫助每一個學生。所以有總結出一些影響學習的個人因素。即非關智力，勤奮，基礎知識等的因素。或者說，我們過去一年基本排除了一些常見的影響學生表現的因素，如課上聽不懂，課後沒人問，沒人討論，學期中不用功，等到學期末臨時抱佛腳，等等。所以我們有機會剝離出幾個我們認為是課堂無法解決的問題。很遺憾，我們確實注意到很多不良因素，以下三個方面尤為明顯：

1. 不少學生不知道為什麼會來數學系，導致學習動力低下。即使提供個人輔導，他們也意願不高。這大概不是課堂能解決的問題。
2. 不少學生對未來一片茫然。沒有期待，也沒有企圖心。這顯然是台灣教育環境中的一個較為嚴重的問題。
3. 校園活動豐富，這本是好事，但比較多學生很明顯不具備平衡生活的能力。本校數學系最重要的活動松數營的最近兩年的執行長都碰巧在本人班上，並被當掉。要知道因為我們提供大量幫助的關係，被當掉是不容易的。

以上三個方面其實是同一件事：**學生缺乏成熟度，缺乏獨立思考的能力。**因為我們過去一直全程帶同一個班兩年，可以說清楚的看見，撇開大學自己的責任（這當然有很多），在中小學與社會環境方面，已經造成一些到大學階段已有點遲，難解決的問題。對比本人過去在美國十來年的任教經驗，美國學生的個性與基礎方面我個人認為不及台灣學生，但成熟度方面則是大勝。這可能是我過去有機會用兩年時間持續近距離接觸同一個班的最大看見。

附件 A B C D

附件 A

注：本附件目的為每名學生可各自自由調整輸出的對應 pdf 檔（見附件 B）。

```
\documentclass[11pt,a4paper]{article}
\usepackage[toc,page]{appendix}
\usepackage{framed}
\usepackage{amsmath}
\usepackage{amssymb}
\usepackage{graphicx}
\usepackage{tcolorbox}
\usepackage{enumerate}
\usepackage{amsthm}
\usepackage{type1cm}

\newcommand{\dislim}{\displaystyle\lim}

\everymath{\displaystyle}
%%%%%%%%%%%%%%%%%%%%%%%%%%%
\addtolength{\voffset}{-2.2cm}
\addtolength{\hoffset}{-2.5cm}
\addtolength{\textwidth}{5cm}
\addtolength{\textheight}{2cm}

%%%%%%%%%%%%%%%%%%%
\def\blue#1{\textcolor[rgb]{0.00,0.00,1.00}{#1}}
\def\red#1{\textcolor[rgb]{1.00,0.00,0.00}{#1}}
\def\purple#1{\textcolor[rgb]{1.00,0.00,0.50}{#1}}
\def\green#1{\textcolor[rgb]{0.00,1.00,0.00}{#1}}


\def\bull#1{\begin{itemize} \item #1 \end{itemize}}
\def\bluecenter#1{\begin{center}
\blue{``#1''}
\end{center}}

\numberwithin{equation}{section}

\def\nobf#1{\noindent \textbf{#1}}
\def\bigno{\bigskip \noindent}
\def\bignobf#1{\bigskip \noindent \textbf{#1}}
\def\rt#1{\sqrt{#1}}
\def\qandq{\quad \text{and} \quad}
\def\qorq{\quad \text{or} \quad}
\def\divides{\bigskip \hrule
\bigno}

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
%\fontsize{10pt}{20pt}
%\selectfont
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
\title{Spivak}
```

```
\date{\today}
\begin{document}
\maketitle

\setcounter{section}{4} %% chapter 5 limit setcounter

\tableofcontents
\newpage
\section{Limit (p.91)}
\subsection{Context}
\begin{enumerate}

\item The concept of a limit is surely the most important, and probably the most
difficult one in all calculus.
\item The goal of this chapter is \bluecenter{the definition of limits,} but we are, once
more, going to begin with a provisional definition;
\bull{what we shall define is not the word ``limit" but the notion of a function
approaching a limit.}

\begin{framed}
$\textbf{PROVISIONAL DEFINITION}$\\ \\
The function $f$ approaches the limit $l$ near $a$, if we can make $f(x)$ as close as
we like to $l$ by requiring that $x$ be
\bull{sufficiently close to, but no equal to, $a$}
\end{framed}

\item Of the six functions graphed in Figure 1, only the first three approach $l$ at $a$.

\begin{center}
\includegraphics[scale=0.4]{figure/chapter5/1.jpg}\\
\end{center}

\item Notice that although $g(a)$ is not defined and $h(a)$ is defined ``the wrong
way", it is still true that $g$ and $h$ approach $l$ near $a$.

\item This is because we explicitly ruled out, in our definition, the necessity of ever
considering the value of the function at a
\bull{--- it is only necessary that $f(x)$ should be close to $l$ for $x$ close to $a$,
but unequal to $a$.}

\item We are simply not interested in the value of $f(a)$, or even in the question of
whether $f(a)$ is defined.

\divides



\item One convenient way of picturing the assertion that $f$ approaches / near $a$ is
provided by a method of drawing functions that was not mentioned in Chapter 4.
```

\item In this method, we draw two straight lines, each representing $\mathbb{R}$,
and arrows from a point x in one, to $f(x)$ in the other.
\item Figure 2 illustrates such a picture for two different functions.
\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/2.jpg}\\

\end{center}
\item Now consider a function f whose drawing looks like Figure 3.
\begin{center}
\includegraphics[scale=0.8]{figure/chapter5/3.jpg}\\

\end{center}
\item Suppose we ask that $f(x)$ be close to l, say within the open interval
B which has been drawn in Figure 3.
\item This can be guaranteed if we consider only the numbers x in the interval
A of Figure 3.
\item (In this diagram we have chosen the largest interval which will work; any
smaller interval containing a could have been chosen instead.)

\item If we choose a smaller interval B' (Figure 4) we will, usually, have to choose
a smaller A',\bull{ but no matter how small we choose the open interval B,}
there is always supposed to be some open interval A which works.

\begin{center}
\includegraphics[scale=0.8]{figure/chapter5/4.jpg}\\
\end{center}
\item A similar pictorial interpretation is possible in terms of the graph of f, but in
this case
\begin{itemize}
\item the interval B must be drawn on the vertical axis, and
\item the set A on the horizontal axis.
\end{itemize}

\item The fact that $f(x)$ is in B when x is in A means that
\bull{ the part of the graph lying over A is contained in the region which is
bounded by the horizontal lines through the end points of B;}
\begin{itemize}
\item compare Figure 5(a), where a valid interval A has been chosen, with Figure
5(b), where A is too large.
\end{itemize}
\begin{center}
\includegraphics[scale=0.4]{figure/chapter5/5.jpg}\\

\end{center}

\item In order to apply our definition to a particular function, let us consider
\begin{equation}
f(x) = x\sin\bigg(\dfrac{1}{x}\bigg)
\end{equation}
(Figure 6).

\begin{center}
\includegraphics[scale=0.6]{figure/chapter5/6.jpg}\\

\end{center}
\item Despite the erratic behavior of this function near 0 it is clear, at least intuitively, that f approaches 0 near 0, and it is certainly to be hoped that our definition will allow us to reach the same conclusion.
\item In the case we are considering, both a and l of the definition are 0, so we must ask if we can get
\begin{equation}
f(x) = x\sin\dfrac{1}{x}
\end{equation}
as close to 0 as desired if we require that x be sufficiently close to 0, but $\neq$ 0.
\item To be specific, suppose we wish to get
\begin{equation}
x\sin\dfrac{1}{x} \text{ within } \dfrac{1}{10} \text{ of } 0.
\end{equation}

\item This means we want
\begin{equation}
-\frac{1}{10} <x \sin \frac{1}{x}< \frac{1}{10}
\end{equation}
or, more sufficiently,
\begin{equation}
\bigg|\sin\bigg(\dfrac{1}{x}\bigg)\bigg|\leq\frac{1}{10}.
\end{equation}
\item Now this is easy.

\item Since
\begin{equation}
\bigg|\sin\frac{1}{x}\bigg|\leq 1\text{, }\forall x \neq 0
\end{equation}

we have
\begin{equation}
\bigg|x\sin\frac{1}{x}\bigg|\leq|x|\text{, }\forall x\neq0
\end{equation}

\item This means that if
\begin{equation}
x<\dfrac{1}{10}\text{ and }x\neq0,
\end{equation}
then \begin{equation}
\bigg|x\sin\dfrac{1}{x}\bigg|<\frac{1}{10};
\end{equation}
\begin{itemize}
\item in other words,
\begin{equation}
x\sin\dfrac{1}{x}\text{ is within }\frac{1}{10} \text{ of 0 }
\end{equation}

```
\item provided that $x$ is within
\begin{equation}
\frac{1}{10} \text{ of }0,\text{ but }\neq 0.
\end{equation}

\end{itemize}

\item There is nothing special about the number $\dfrac{1}{10}$;

\begin{itemize}
\item it is just as easy to guarantee that
\begin{equation}
|f(x)-0|<\frac{1}{100}
\end{equation}
\item simply require that
\begin{equation}
\dfrac{1}{100} \text{, but } x \neq 0.
\end{equation}

\end{itemize}

\item In fact, if we take any positive number $\varepsilon$ we can make
\begin{equation}
|f(x)-0|<\varepsilon
\end{equation}
simply by requiring that $|x|<\varepsilon$, and $x\neq0$

\item For the function $$ f(x) = x^{2}\sin\dfrac{1}{x}$$ (Figure 7) it seems even
clearer that $f$ approaches 0 near 0.
\begin{center}
\includegraphics[scale=0.6]{figure/chapter5/7.jpg}\\
\end{center}


\item If, for example, we want

\begin{equation}
|x^{2}\sin\frac{1}{x}|<\frac{1}{10}
\end{equation}

then we certainly need only require that
\begin{equation}
|x|<\frac{1}{10}\text{ and }x\neq0,
\end{equation}
since this implies that $|x^{2}|<\dfrac{1}{100}$ and consequently

\begin{equation}|x^{2}\sin\dfrac{1}{x}|\leq|x^{2}|<\dfrac{1}{100}<\frac{1}{10}\en
d{equation}
```

```
\item (We could do even better, and allow
\begin{equation}
|x| < \dfrac{1}{\sqrt{10}}\text{ and }x\neq0,
\end{equation}

but there is no particular virtue in being as economical as possible.)
\item In general, if $\varepsilon>$ 0, to ensure that
\begin{equation}
|x^{2}\sin\dfrac{1}{x}|<\varepsilon
\end{equation}
we need only require that
\begin{equation}|x|<\varepsilon\text{ and }x\neq0,\end{equation}

provided that $\varepsilon\leq1$.
\item If we are given an $\varepsilon$ which is greater than 1
\begin{itemize}
\item (it might be, even thought it is “small” $\varepsilon$'s which are of interest),
\item then it does not suffice to require that $|x|<\varepsilon$,
\item but it certainly suffices to require that $|x| < 1$ and $x\neq0$.
\end{itemize}


\item As a third example, consider the function
\begin{equation}
f(x) = \sqrt{|x|}\sin\bigg(\dfrac{1}{x}\bigg)\text{ (Figure 8).}
\end{equation}

\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/8.jpg}\\
\end{center}
\item In order to make
\begin{equation}
\bigg|\sqrt{|x|}\sin\dfrac{1}{x}\bigg| < \varepsilon
\end{equation}
we can require that

\begin{equation}|x|<\varepsilon^{2} \text{ and } x\neq0\end{equation}(the algebra is
left to you).

\item Finally, let us consider the function $f(x) = \sin \dfrac{1}{x}$ (Figure 9).
\begin{center}
\includegraphics[scale=0.4]{figure/chapter5/9.jpg}\\
\end{center}
\item For this function it is $false$ that $f$ approaches 0 near 0.
\item This amounts to saying that it is not true for every number $\varepsilon>0$ that
we can get
\begin{equation}
|f(x)-0| < \varepsilon
\end{equation}
by choosing $x$ sufficiently small, and $\neq0$.
```

```
\item To show this we simply have to find one $\varepsilon>0$ for which the condition
\begin{equation}
|f(x)-0| < \varepsilon
\end{equation}
cannot be guaranteed, no matter how small we require $|x|$ to be.
\item In fact, $\varepsilon = \frac{1}{2}$ will do:
\begin{itemize}
\item it is impossible to ensure that $|f(x)|<\frac{1}{2}$ no matter how small we require $|x|$ to be;
\item for if $A$ is any interval containing 0, there is some number $x = \dfrac{1}{(90+360n)}$ which is in this interval, and for this $x$ we have $f(x) = 1$.
\end{itemize}

\item This same argument can be used (Figure 10) to show that $f$ does not approach any number near 0.
\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/10.jpg}\\
\end{center}
\item To show this we must again find, for any particular number $l$, some number $\varepsilon>0$ so that
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
is $not$ true, no matter how small x is required to be.
\item The choice $\varepsilon = \frac{1}{2}$ works for any number $l$; that is, no matter how small we require $|x|$ to be, we cannot ensure that
\begin{equation}
|f(x)-l|<\frac{1}{2} .
\end{equation}
\item The reason is, that for any interval $A$ containing 0 there is some
\begin{equation}
x_{1} = \dfrac{1}{(90+360n)}
\end{equation}
in this interval, so that
\begin{equation}
f(x_{1})=1
\end{equation}
and also some
\begin{equation}
x_{2} = \dfrac{1}{(270+360m)}
\end{equation}
in this interval, so that
\begin{equation}
f(x_{2})=-1
\end{equation}
```

\item But the interval from $l-\frac{1}{2}$ to $l+\frac{1}{2}$ cannot contain both –1 and 1, since its total length is only 1; so we cannot have
\begin{equation}|1-l|<\frac{1}{2}\text{ and also }|-1-l|<\frac{1}{2}\end{equation}
no mater what $ l$ is

\item The phenomenon exhibited by $f(x)=\sin\dfrac{1}{x}$ near 0 can occur in many ways.
\item If we consider the function
\begin{equation}f(x)=
\begin{cases}
0\text{, }x \text{ irrational}\\
1\text{, }x \text{ rational},
\end{cases}
\end{equation}

then no matter what a is, f does not approach any number l near a.
\item In fact, we cannot make
\begin{equation}
|f(x)-l|<\frac{1}{4}
\end{equation}
no matter how close we bring x to a,
\begin{itemize}
\item because in any interval around a there are numbers x with $f(x)=0$,
\item and also numbers x with $f(x)=1$, so that we would need
\begin{equation}
|0-l|<\frac{1}{4}\text{ and also }|1-l|<\frac{1}{4}.
\end{equation}

\end{itemize}

\item An amusing variation on this behavior is presented by the function shown in Figure 11:
\begin{equation}f(x)=
\begin{cases}
x,\text{ }x \text{ rational}\\
1,\text{ }x \text{ irrational},
\end{cases}\end{equation}

\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/11.jpg}\\
\end{center}

\item The behavior of this function is ``opposite" to that of
\begin{equation}
g(x) = \sin \dfrac{1}{x}
\end{equation}
it approaches 0 at 0, but does not approach any number at a, if $a\neq$ 0.
\item By now you should have no difficulty convincing yourself that this is true.

\item As a contrast to the functions considered so far, which have been quite pathological, we will now examine some of the simplest functions.

\item If $f(x) = c$, then f approaches c near a, for every number a.
\item In fact, to ensure that
\begin{equation}
|f(x)-c|<\varepsilon
\end{equation}
one does not need to restrict x to be near a at all; the condition is automatically satisfied\\ (Figure 12).

\begin{center}
\includegraphics{figure/chapter5/12.jpg}\\
\end{center}

\item As a slight variation, let f be the function shown in Figure 13:
\begin{equation}f(x)=
\begin{cases}
-1,\text{ }x<0\\
\text{ }1\text{ },\text{ }x>0\\
\end{cases}
\end{equation}

\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/13.jpg}\\
\end{center}

\item If $a>0$, then f approaches l near a: indeed, to ensure that
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
it certainly suffices to require that
\begin{equation}
|x-a|<a,
\end{equation} since this implies
\begin{equation}
-a<x-a
\end{equation}
\begin{equation}
\text{or }0<x
\end{equation}

so that $f(x) = 1$.
\item Similarly, if $b < 0$, then f approaches -1 near b: to ensure that
\begin{equation}
|f(x)–(-1)|<\varepsilon
\end{equation}
it suffices to require that $|x-b|<-b$.
\item Finally, as you may easily check, f does not approach any number near 0.

\item The function $f(x) = x$ is easily dealt with.
\item Clearly f approaches a near a: to ensure that
\begin{equation}
|f(x)-a|<\varepsilon
\end{equation}
we just have to require that
\begin{equation}
|x–a|<\varepsilon.
\end{equation}

\item The function $f(x)=x^2$ requires a little more work.
\item To show that f approaches a^2 near a , we must decide how to ensure that
\begin{equation}
|x^2-a^2|<\varepsilon.
\end{equation}

\item Factoring looks like the most promising procedure: we want
\begin{equation}
|x-a|\cdot|x+a|<\varepsilon
\end{equation}

\item Obviously the factor $|x+a|$ is the one that will cause trouble.
\item On the other hand, there is no need to make $|x + a|$ particularly small;\bull{ as long as we know some bound on the values of $|x + a|$ we will be in good shape.}
\item For example, if
\begin{equation}
|x+a| < 1,000,000,
\end{equation}
then we will just need to require that
\begin{equation}
|x-a| < \dfrac{\varepsilon}{10000000}.
\end{equation}
\item Therefore, to begin with, let us require that $|x-a|<1$
\bull{(any positive number other than I would do just as well);}
\bull{ presumably this will ensure that x is not too large, and consequently that $|x+ a|$ is not too large.}
\item As a matter of fact, Problem 1-12 shows that

\begin{equation}
|x|-|a|\leq|x-a|<1,
\end{equation}
so
\begin{equation}
|x|<1+|a|,
\end{equation}
and consequently
\begin{equation}|x+a|\leq|x|+|a|<2|a|+1\end{equation}.
\item Now we need only the additional requirement that
\begin{equation}

```
|x-a|<\dfrac{\varepsilon}{(2|a|+1)}.
\end{equation}
\item In other words, if
\begin{equation}
|x-a|<\min\left(1,\frac{\varepsilon}{2|a|+1}\right),
\end{equation}
then
\begin{equation}
|x^2-a^2|<\varepsilon.
\end{equation}

\item Naturally,
\begin{equation}
\min\left(1,\dfrac{\varepsilon}{(2|a|+1)}\right)
\end{equation}
will just be
\begin{equation}
\dfrac{\varepsilon}{(2|a|+1)}
\end{equation}
for small $\varepsilon$.
\item Precisely the same sort of trick will show that if
\begin{equation}
f(x)=x^3,
\end{equation}
then $f$ approaches $a^3$ near $a$.
\item In fact, if
\begin{equation}
|x-a|<\min\left(1,\frac{\varepsilon}{(1+|a|)^{2}+|a|(1+|a|)+|a|^{2}}\right),
\end{equation}
then
\begin{equation}
|x^3-a^3|<\varepsilon.
\end{equation}
\item The proof of this assertion will show where the weird denominator comes from:

\item If $|x-a|<1$, then $|x|<|a|+1$, and consequently

\begin{equation}
\begin{aligned}
|x^2+ax+a^2| &\leq |x|^2+|a|\cdot|x|+|a|^2 \\
&<(1+|a|)^2+|a|(1+|a|)+|a|^2
\end{aligned}
\end{equation}

\item Therefore

\begin{equation}
\begin{aligned}
|x^3-a^3|&=|x-a|\cdot|x^2+ax+a^2|\\
```

&<\frac{\varepsilon}{(1+|a|)^2+|a|(1+|a|)+|a|^2}\cdot\left[(1+|a|)^2+|a|(1+|a|)+|a|^2\right]\\
&=\varepsilon
\end{aligned}
\end{equation}

\item The time has now come to point out that of the many demonstrations about limits which we have given, not one has been a real proof.
\item The fault lies not with our reasoning, but with our definition.
\item If our provisional definition of a function was open to criticism, our provisional definition of approaching a limit is even more vulnerable.
\item This definition is simply not sufficiently precise to be used in proofs.
\item It is hardly clear how one ``makes'' $f(x)$ close to $1$ \bull{(whatever ``close'' means) } by ``requiring'' $x$ to be sufficiently close to $a$ \bull{(however close ``sufficiently'' close is supposed to be).}
\item Despite the criticisms of our definition you may feel \bull{(I certainly hope you do)} that our arguments were nevertheless quite convincing.
\item In order to present any sort of argument at all, we have been practically forced to invent the real definition.
\item It is possible to arrive at this definition in several steps, each one clarifying some obscure phrase which still remains.
\item Let us begin, once again, with the provisional definition:
\begin{itemize}
\item The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to, a.
\end{itemize}

\item The very first change which we made in this definition was to note that making $f(x)$ close to l meant making
\begin{equation}
|f(x)-l|
\end{equation} small, and similarly for x and a:
\begin{itemize}
\item The function f approaches the limit l near a, if we can make $|f(x)-l|$ as small as we like by requiring that $|x - a|$ be sufficiently small, and $x\neq a$.
\end{itemize}

\item The second, more crucial, change was to note that making
\begin{equation}
|f(x)-l|
\end{equation}
``as small as we like'' means making
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
for any $\varepsilon>0$ that happens to be given us:
\begin{itemize}
\item The function f approaches the limit l near a, if for every number $\varepsilon > 0$ we can make

\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
by requiring that $|x - a|$ be sufficiently small, and $x\neq a$.
\end{itemize}

\item There is a common pattern to all the demonstrations about limits which we have given.
\item For each number $\varepsilon > 0$ we found some other positive number, δ say, with the property that if
\begin{equation}
x\neq a\text{ and }|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon.
\end{equation}
\item For the function
\begin{equation}
f(x) = x \sin \bigg(\dfrac{1}{x}\bigg)
\end{equation}
\bull{
(with $a = 0$, $l=0$), the number δ was just the number ε;}
\bull{ for
\begin{equation}
f(x) = \sqrt{|x|}\sin\bigg(\dfrac{1}{x}\bigg),
\end{equation} it was ε^2 for
\begin{equation}
f(x) = x^2
\end{equation}
it was the minimum of
\begin{equation}
1\text{ and }\dfrac{\varepsilon}{2|a|+1}.
\end{equation}
}
\item In general, it may not be at all clear how to find the number δ, given ε, but it is the condition
\begin{equation}
|x-a| < \delta
\end{equation}
which expresses how small ``sufficiently" small must be:
\begin{itemize}
\item The function f approaches the limit l near a, if for every $\varepsilon > 0$
\item there is some $\delta >0$ such that, for all x, if $|x-a| < \delta$ and $x\neq a$, then $|f(x)-1| <\varepsilon$.
\end{itemize}

\item This is practically the definition we will adopt.

\item We will make only one trivial change, nothing that ``$|x-a|<\delta$ and $x \neq a$" can just as well be expressed ``$0<|x-a|<\delta$".

\begin{framed}
DEFINITION

The function f approaches the limit l near a means:
\bull{ for every $\varepsilon > 0$ there is some $\delta >0$ such that, for all x, if
\begin{equation}
0 < |x - a| < \delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon.
\end{equation}
}

\end{framed}

\item This definition is so important \bull{(everything we do from now on depends on it)} that proceeding any further without knowing it is hopeless.
\item If necessary memorize it, like a poem!
\item That, at least, is better than stating it incorrectly;
\bull{ if you do this you are doomed to give incorrect proofs.}
\item A good exercise in giving correct proofs is to review every fact already demonstrated about functions approaching limits, giving real proofs of each.
\item This requires writing down the correct definition of what you are proving, but not much more-all the algebraic work has been done already.
\item When proving that f does not approach l at a, be sure to negate the definition correctly:

\item If it is not true that
\begin{itemize}
\item for every $\varepsilon>0$
\item there is some $\delta>0$ such that, for all x if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
\end{itemize}
\begin{itemize}
\item then there is some $\varepsilon>0$ such that for every $\delta>0$
\item there is some x which satisfies
\begin{equation}
0<|x-a|<\delta
\end{equation}

but not
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
\end{itemize}

\item Thus, to show that the function
\begin{equation}
f(x) = \sin \dfrac{1}{x}
\end{equation}
does not approach 0 near 0, we consider $\varepsilon = \frac{1}{2}$ and note that for every $\varepsilon> 0$ there is some x with
\begin{equation}
0< |x -0| <\delta
\end{equation}
but not
\begin{equation}
\bigg|\sin\dfrac{1}{x}- 0\bigg|<\frac{1}{2}
\end{equation}
-namely, an x of the form
\begin{equation}
\dfrac{1}{(90+360n)},
\end{equation}
where n is so large that
\begin{equation}
\dfrac{1}{(90 + 360n)} < \delta.
\end{equation}
\item As an illustration of the use of the definition of a function approaching a limit, we have reserved the function shown in Figure 14, a standard example, but one of the most complicated:
\begin{equation}
f(x)=
\begin{cases}
0\text{ , }x\text{ irrational, }0<x<1\\
\dfrac{1}{q},\text{ }x=\dfrac{p}{q}\text{ in lowest terms, }0<x<1
\end{cases}
\end{equation}
\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/14.jpg}\\
\end{center}
\item (Recall that $\dfrac{p}{q}$ is in lowest terms if p and q are integers with no common factor and $q > 0$.)

For any number a, with $0 < a < 1$, the function f approaches 0 at a.
\item To prove this, consider any number $\varepsilon > 0$.
\item Let n be a natural number so large that $\dfrac{1}{n} \leq \varepsilon$.
\item Notice that the only numbers x for which $|f(x) - 0| < \varepsilon$ could be false are:

\begin{quote}

\begin{equation}
\frac{1}{2};\frac{1}{3},\frac{2}{3};\frac{1}{4},\frac{3}{4};\frac{1}{5},\frac{2}{5
},\frac{3}{5},\frac{4}{5};...;\frac{1}{n},...,\frac{n-1}{n}
\end{equation}
\end{quote}

\item(If a is rational, then a might be one of these numbers.)
\item However many of these numbers there may be, there are, at any rate, only finitely many.
\item Therefore, of all these numbers, one is closest to a; that is,
\begin{equation}
\bigg|\dfrac{p}{q}- a\bigg|
\end{equation}
is smallest for one $\dfrac{p}{q}$ among these numbers.
\item (If a happens to be one of these numbers, then consider only the values
$ \left|\dfrac{p}{q}- a\right|$ for $\dfrac{p}{q}\neq a$.)
\item This closest distance may be chosen as the δ.
\item For if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then x is not one of

\begin{quote}
\begin{equation}\frac{1}{2},...,\frac{n-1}{n}\end{equation}
\end{quote}

and therefore $|f(x) - 0| <\varepsilon$ is true.
\item This completes the proof.
\item Note that our description of the δ which works for a given ε is completely adequate---there is no reason why we must give a formula for δ in terms of ε.

\item Armed with our definition, we are now prepared to prove our first theorem;
\bull{ you have probably assumed the result all along, which is a very reasonable thing to do.}
\item This theorem is really a test case for our definition:\bull{ if the theorem could not be proved, our definition would be useless.}

\begin{framed}
THEOREM 1

A function cannot approach two different limit near a.
\bull{ In other words, if f approaches l near a, and f approaches m near a, then $l=m$.}

\end{framed}

PROOF

\item Since this is our first theorem about limits it will certainly be necessary to translate the hypotheses according to the definition.
\item Since f approaches l near a, we know that for any $\varepsilon > 0$ there is some number $\delta_1>0$ such that, for all x, if
\begin{equation}
0 < |x-a| < \delta_1,
\end{equation}
then
\begin{equation}
|f(x) -1| < \varepsilon.
\end{equation}
\item We also know, since f approaches m near a, that there is some $\delta_2 > 0$ such that, for all x, if
\begin{equation}
0 < |x-a| < \delta_2,
\end{equation}
then
\begin{equation}
|f(x) - m| < \varepsilon.
\end{equation}
\item We have had to use two numbers, δ_1 and δ_2, since there is no guarantee that the δ which works in one definition will work in the other.
\item But, in fact, it is now easy to conclude that for any $\varepsilon > 0$ there is some $\delta >0$ such that, for all x, if
\begin{equation}
0 < |x - a| < \delta,
\end{equation}
then
\begin{equation}
|f(x) -l| < \varepsilon \text{ and } |f(x)-m| < \varepsilon;
\end{equation}
we simply choose
\begin{equation}
\delta =\min(\delta_1, \delta_2).
\end{equation}
\item To complete the proof we just have to pick a particular $\varepsilon > 0$ for which the two conditions
\begin{equation}|f(x) -l| <\varepsilon \text{ and } |f(x) - m| <\varepsilon\end{equation}
cannot both hold, if $l \neq m$.
\item The proper choice is suggested by Figure 15.
\begin{center}
\includegraphics[scale=0.6]{figure/chapter5/15.jpg}\\
\end{center}
\item If $l \neq m$, so that
\begin{equation}
|l-m|>0,
\end{equation}
we can choose $\dfrac{|l - m|}{2}$ as our ε.
\item It follows that there is a $\delta >0$ such that, for all x, if

\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\frac{|l-m|}{2} \text{ and }|f(x)-m|<\frac{|l-m|}{2}.
\end{equation}

\item This implies that for $0<|x-a|<\delta$ we have

\begin{equation}
\begin{aligned}
|l-m|
&=|l-f(x)+f(x)-m|\\
&\leq |l-f(x)|+|f(x)-m|\\
&<\frac{|l-m|}{2}+\frac{|l-m|}{2}\\
&=|l-m|
\end{aligned}
\end{equation}

a contradiction. $\blacksquare$

\item The number l which f approaches near a is denoted by
$\displaystyle\lim_{x \to a}f(x)$ \bull{(read: the limit of $f(x)$ as x approaches
a).}
\item This definition is possible only because of Theorem 1, which ensures that
$\displaystyle\lim_{x \to a}f(x)$ never has to stand for two different numbers.
\item The equation
\begin{equation}\displaystyle\lim_{x \to a}f(x) = l\end{equation}
has exactly the same meaning as the phrase
\begin{equation}f\text{ approaches } l \text{ near }a.\end{equation}
\item The possibility still remains that f does not approach l near a, for any
l, so that $\displaystyle\lim_{x \to a}f(x)=l$ is false for every number l.
\item This is usually expressed by saying that "$\displaystyle\lim_{x \to a} f(x)$ does
not exist."
\item Notice that our new notation introduces an extra, utterly irrelevant letter x,
\bull{ which could be replaced by t, y, or any other letter which does not already
appear — the symbols}
\begin{equation}\displaystyle\lim_{x \to a}f(x),\hspace{1cm}\displaystyle\lim_{t \to
a}f(t),\hspace{1cm}\displaystyle\lim_{y \to a}f(y),\end{equation}
\bull{all denote precisely the same number, which depends on f and a, and has
nothing to do with x, t, or y.}
\bull{(these letters, in fact, do not denote anything at all).}
\item A more logical symbol would be something like $\displaystyle\lim_{a}f$, but
this notation, despite its brevity, is so infuriatingly rigid that almost no one has
seriously tried to use it.
\item The notation $\displaystyle\lim_{x \to a} f(x)$ is much more useful because a
function f often has no simple name, even though it might be possible to express
$f(x)$ by a simple formula involving x.
\item Thus, the short symbol

\begin{equation}\displaystyle\lim_{x \to a}(x^2+\sin x)\end{equation}
could be paraphrased only by the standard symbolism is illustrated by the expressions
\begin{equation}
\begin{aligned}
\displaystyle\lim_{x \to a}x+t^3\\
\displaystyle\lim_{t \to a}x+t^3
\end{aligned}
\end{equation}
\item The first means the number which f approaches near a when
\begin{equation}f(x)=x+t^3,\hspace{0.5cm}\text{for all }x\end{equation}
the second mean the number which f approaches near a when
\begin{equation}f(t)=x+t^3,\hspace{0.5cm}\text{for all }t\end{equation}
\item You should have little difficult (especially if you consult Theorem 2) proving that
\begin{equation}
\begin{aligned}
\displaystyle\lim_{x \to a}x+t^3 = a+t^3\\
\displaystyle\lim_{t \to a}x+t^3 = x+a^3
\end{aligned}
\end{equation}

\item These examples illustrate the main advantage of our notation, which is its flexibility.
\item In fact, the notation $\displaystyle\lim_{x \to a}f(x)$ is so flexible that there is some danger of forgetting what it really means.
\item Here is a simple exercise in the use of this notation, which will be important later: first interpret precisely, and then prove the equality of the expressions
\begin{equation}
\displaystyle\lim_{x \to a}f(x)\hspace{0.3cm}\text{and}\hspace{0.3cm}\displaystyle\lim_{h \to 0}f(a+h)
\end{equation}
\item An important part of this chapter is the proof of a theorem which will make it easy to find many limits.
\item The proof depends upon certain properties of inequalities and absolute values, hardly surprising when one considers the definition of limit.
\item
\begin{itemize}
\item Although these facts have already been stated in Problems 1-20, 1-21, and 1-22, because of their importance they will be presented once again,
\item in the form of a lemma (a lemma is an auxiliary theorem, a result that justifies its existence only by virtue of its prominent role in the proof of another theorem).
\end{itemize}

\item
The lemma says, roughly, that if x is close to x_0, and y is close to y_0, then $x + y$ will be close to $x_0 + y_0$, and xy will be close to $x_0 y_0$, and $\dfrac{1}{y}$ will be close to $\dfrac{1}{y_0}$.
\item This intuitive statement is much easier to remember than the precise estimates of the lemma,

\bull{ and it is not unreasonable to read the proof of Theorem 2 first,}
\bull{ in order to see just how these estimates are used.}

\begin{framed}
LEMMA

\begin{enumerate}
\item If
\begin{equation}
|x-x_0|<\frac{\varepsilon}{2}\text{ and }|y-y_0|<\frac{\varepsilon}{2}
\end{equation}
then
\begin{equation}
|(x+y)-(x_0+y_0)|<\varepsilon
\end{equation}

\item If
\begin{equation}
|x-x_0|<\min\left(1,\frac{\varepsilon}{2(|y_0|+1)}\right)\text{ and }|y-
y_0|<\frac{\varepsilon}{2(|x_0|+1)},
\end{equation}
then
\begin{equation}
|xy-x_0 y_0|<\varepsilon.
\end{equation}

\item If
\begin{equation}
y_0\neq 0 \text{ and }|y-
y_0|<\min\left(\frac{|y_0|}{2},\frac{\varepsilon|y_0|^2}{2}\right),
\end{equation}
then
\begin{equation}
y\neq 0 \text{ and }\left|\frac{1}{y}-\frac{1}{y_0}\right|<\varepsilon.
\end{equation}
\end{enumerate}
\end{framed}

PROOF

\begin{enumerate}
\item
\begin{equation}
\begin{aligned}
|(x+y)-(x_0+y_0)|&=|(x-x_0)+(y-y_0)|\\
&\leq |x-x_0|+|y-y_0|\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&=\varepsilon.
\end{aligned}
\end{equation}

```
\item
\begin{enumerate}[1.]
\item Since $|x-x_0| < 1$ we have
\begin{equation}
|x|-|x_0| \leq |x-x_0|<1,
\end{equation}
so that
\begin{equation}
|x|<1+|x_0|.
\end{equation}
\item Thus
\begin{equation}
\begin{aligned}
|xy-x_0 y_0|&=|x(y-y_0)+y_0(x-x_0)|\\
&\leq |x|\cdot|y-y_0|+|y_0|\cdot|x-x_0|\\
&<(1+|x_0|)\cdot\frac{\varepsilon}{2(|x_0|+1)}+|y_0|\cdot\frac{\varepsilon}{2(|y_0|
+1)}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&=\varepsilon.
\end{aligned}
\end{equation}
\end{enumerate}

\item
\begin{enumerate}[1.]
\item We have
\begin{equation}
|y_0|-|y| \leq |y-y_0|<\frac{y_0}{2},
\end{equation}
so $|y|>\dfrac{|y_0|}{2}$.
\item In particular, $y \neq 0$,and
\begin{equation}
\frac{1}{|y|}<\frac{2}{|y_0|}.
\end{equation}
\item Thus
\begin{equation}
\left|\frac{1}{y}-\frac{1}{y_0}\right| = \frac{|y_0-y|}{|y|\cdot|y_0|} < \frac{2}{|y_0|}
\cdot \frac{1}{|y_0|} \cdot \frac{\varepsilon|y_0|^2}{2} = \varepsilon.\blacksquare
\end{equation}
\end{enumerate}

\end{enumerate}
\begin{framed}
THEOREM 2
\begin{enumerate}[1.]
\item If $\displaystyle\lim_{x \to a}f(x)=l$ and $\displaystyle\lim_{x \to a}g(x)=m$,
then
\begin{equation}
\begin{aligned}
```

```
&(1)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f+g)(x)=l+m;\\
&(2)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f \cdot g)(x)=l \cdot m.
\end{aligned}
\end{equation}
\item Moreover, if $m \neq 0$, then
\begin{equation}
(3)\hspace{0.5cm} \displaystyle\lim_{x \to a}\left(\frac{1}{g}\right)(x) =
\dfrac{1}{m}.
\end{equation}
\end{enumerate}
\end{framed}

PROOF:
\begin{enumerate}[1.]
\item The hypothesis means that for every $\varepsilon>0$ there are
$\delta_1,\delta_2>0$ such that, for all $x$, if
\begin{equation}
0<|x-a| <\delta_1,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon
\end{equation}
and if
\begin{equation}
0<|x-a| <\delta_2
\end{equation}
then
\begin{equation}
|g(x)-m|<\varepsilon
\end{equation}
\item This means (since, after all, $\frac{\varepsilon}{2}$ is also a positive number)
that there are $\delta_1,\delta_2>0$ such that, for all $x$, if
\begin{equation}
0<|x-a| <\delta_1
\end{equation}
then
\begin{equation}
|f(x)-l|<\frac{\varepsilon}{2}
\end{equation}
and if
\begin{equation}
0<|x-a| <\delta_2,
\end{equation}
then
\begin{equation}
|g(x)-m|<\frac{\varepsilon}{2}.
\end{equation}
\item Now let
\begin{equation}
```

```
\delta=\min(\delta_1,\delta_2).
\end{equation}
\item If
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
0<|x-a|<\delta_1
\end{equation}
and
\begin{equation}
0<|x-a|<\delta-2
\end{equation}
are both true, so both
\begin{equation}
|f(x)-l|<\frac{\varepsilon}{2}\text{ and }|g(x)-m|<\frac{\varepsilon}{2}.
\end{equation}
are true.
\item But by part (1) of the lemma this implies that
\begin{equation}
|(f+g)(x)-(l-m)|<\varepsilon.
\end{equation}
\item This proves (1).
\item To prove (2) we proceed similarly, after consulting part (2) of the lemma,
$\varepsilon>0$
\bull{ there are $\delta_1,\delta_2>0$ such that, for all $x$,}
if
\begin{equation}
0<|x-a|<\delta_1,
\end{equation}
then
\begin{equation}
|f(x)-l|<\min\left(1,\dfrac{\varepsilon}{2(|m|+1)}\right),
\end{equation}
and if
\begin{equation}
0<|x-a|<\delta_2,
\end{equation}
then
\begin{equation}
|g(x)-m|<\frac{\varepsilon}{2(|l|+1)}.
\end{equation}
\item Again let
\begin{equation}
\delta = \min(\delta_1,\delta_2).
\end{equation}
\item If $0<|x-a|<\delta$, then
\begin{equation}
|f(x)-l|<\min\left(1,\frac{\varepsilon}{2(|m|+1)}\right)
```

\end{equation}
and
\begin{equation}
|g(x)-m|<\frac{\varepsilon}{2(|l|+1)}.
\end{equation}
\item So, by the lemma,
\begin{equation}
|(f \cdot g)(x)-l \cdot m|<\varepsilon,
\end{equation}
and this proves (2).

\item Finally, if $\varepsilon>0$ there is a $\delta>0$ such that, for all x, if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
|g(x)-m|<\min\left(\frac{|m|}{2},\frac{\varepsilon|m|^2}{2}\right).
\end{equation}
\end{enumerate}

\item But according to part (3) of the lemma this means, first, that $g(x)\neq 0$, so $\dfrac{1}{g}(x)$ makes sense, and second that

\begin{equation}
|\frac{1}{g}(x)-\frac{1}{m}|<\varepsilon.
\end{equation}
\item This proves (3).
\begin{framed}
THEOREM 2
\begin{enumerate}[1.]
\item If $\displaystyle\lim_{x \to a}f(x)=l$ and $\displaystyle\lim_{x \to a}g(x)=m$, then
\begin{equation}
\begin{aligned}
&(1)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f+g)(x)=l+m;\\
&(2)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f \cdot g)(x)=l \cdot m.
\end{aligned}
\end{equation}
\item Moreover, if $m \neq 0$, then
\begin{equation}
(3)\hspace{0.5cm} \displaystyle\lim_{x \to a}\left(\frac{1}{g}\right)(x) = \frac{1}{m}.
\end{equation}
\end{enumerate}
\end{framed}

\item Using Theorem 2 we can prove, trivially, such facts as

\begin{equation} \lim_{x \to a} \frac{x^3+7x^5}{x^2+1} =
\frac{a^3+7a^5}{a^2+1},\end{equation}
without going through the laborious process of finding a δ, given an
ε.
\item We must begin with
\begin{equation} \lim_{x \to a} 7 = 7, \end{equation}
\begin{equation} \lim_{x \to a} 1 = 1, \end{equation}
\begin{equation} \lim_{x \to a} x = a, \end{equation}
but these are easy to prove directly.
\item If we want to find the δ, however, the proof of Theorem 2 amounts to a
prescription for doing this.
\item Suppose, to take a simpler example, that we \textit{want} to find a δ such
that, for all x, if
\begin{equation}
0<|x-a|<\delta
\end{equation}
then
\begin{equation}
|x^2+x-(a^2+a)|<\varepsilon.
\end{equation}
\begin{framed}
THEOREM 2
\begin{enumerate}[1.]
\item If $\displaystyle\lim_{x \to a}f(x)=l$ and $\displaystyle\lim_{x \to a}g(x)=m$,
then
\begin{equation}
\begin{aligned}
&(1)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f+g)(x)=l+m;\\
&(2)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f \cdot g)(x)=l \cdot m
\end{aligned}
\end{equation}
\item Moreover, if $m \neq 0$, then
\begin{equation}
(3)\hspace{0.5cm} \displaystyle\lim_{x \to a}\left(\frac{1}{g}\right)(x) =
\frac{1}{m}
\end{equation}
\end{enumerate}
\end{framed}

\item Consulting the proof of Theorem 2(1), we see that we must first find
δ_1 and $\delta_2>$ 0 such that, for all x, if
\begin{equation}
0<|x-a|<\delta_1,
\end{equation}
then
\begin{equation}
|x^2-a^2|<\frac{\varepsilon}{2}
\end{equation}
if
\begin{equation}

0<|x-a|<\delta_2,
\end{equation}
then
\begin{equation}
|x-a|<\frac{\varepsilon}{2}.
\end{equation}

\item Since we have already given proofs that $\displaystyle\lim_{x \to
a}{x^2}=a^2$ and $\displaystyle\lim_{x \to a}{x}=a$, we know how to do this:
\begin{equation}\delta_1\text{ =
min}\left(1,\frac{\frac{\varepsilon}{2}}{2|a|+1}\right),\text{ }
\delta_2 = \frac{\varepsilon}{2}\end{equation}
\item Thus we can take
\begin{equation}
\begin{aligned}
\delta &= \text{min }(\delta_1,\delta_2)\\
&=\text{min}\left(\text{min}\left(1,\frac{\frac{\varepsilon}{2}}{2|a|+1}\right),\frac{
\varepsilon}{2}\right).
\end{aligned}
\end{equation}
\item If a $\neq$ 0, the same method can be used to find a $\delta>$ 0 such that, for
all x, if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
\left|\frac{1}{x^2}-\frac{1}{a^2}\right|<\varepsilon.
\end{equation}
\begin{framed}
THEOREM 2
\begin{enumerate}[1.]
\item If $\displaystyle\lim_{x \to a}f(x)=l$ and $\displaystyle\lim_{x \to a}g(x)=m$,
then
\begin{equation}
\begin{aligned}
&(1)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f+g)(x)=l+m;\\
&(2)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f \cdot g)(x)=l \cdot m.
\end{aligned}
\end{equation}
\item Moreover, if $m \neq 0$, then
\begin{equation}
(3)\hspace{0.5cm} \displaystyle\lim_{x \to a}\left(\frac{1}{g}\right)(x) =
\frac{1}{m}.
\end{equation}
\end{enumerate}
\end{framed}

\item The proof of Theorem 2(3) shows that the second condition will follow if we
find a $\delta >$ 0 such that, for all x, if

\begin{equation}
0<|x-a|<\delta,
\end{equation}
\begin{equation}
|x^2-
a^2|<\text{min}\left(\frac{|a|^2}{2},\frac{\varepsilon|a|^4}{2}\right).\end{equation}
\item Thus we can take
\begin{equation}\delta=\min\left(1,\frac{\min(\frac{|a|^2}{2},\frac{\varepsilon|a|^4}{2})}{2|a|+1}\right)\end{equation}
\item Naturally, these complicated expressions for δ can be simplified considerably, after they have been derived.
\begin{framed}
THEOREM 2
\begin{enumerate}[1.]
\item If $\displaystyle\lim_{x \to a}f(x)=l$ and $\displaystyle\lim_{x \to a}g(x)=m$, then
\begin{equation}
\begin{aligned}
&(1)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f+g)(x)=l+m;\\
&(2)\hspace{0.5cm} \displaystyle\lim_{x \to a}(f \cdot g)(x)=l \cdot m.
\end{aligned}
\end{equation}
\item Moreover, if $m \neq 0$, then
\begin{equation}
(3)\hspace{0.5cm} \displaystyle\lim_{x \to a}\left(\frac{1}{g}\right)(x) =
\frac{1}{m}.
\end{equation}
\end{enumerate}
\end{framed}

\item
One technical detail in the proof of Theorem 2 deserves some discussion.
\item In order for $\displaystyle\lim_{x \to a}{f(x)}$ to be defined it is, as we know, not necessary for f to be defined at a, nor is it necessary for f to be defined at all points $x\neq a$.
\item However, there must be some $\delta>$ 0 such that $f(x)$ is defined for x satisfying
\begin{equation}
0<|x-a|<\delta.
\end{equation}
\item Otherwise the clause ``if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon"
\end{equation}
would make no sense at all, since the symbol $f(x)$ would make no sense for some x's.

\item If f and g are two functions for which the definition makes sense, it is easy to see that the same is true for $f + g$ and $f \cdot g$
\item But this is not so clear for $\dfrac{1}{g}$, since $\dfrac{1}{g}$ is undefined for x with $g(x) = 0$.
\item However, this fact gets established in the proof of Theorem 2(3).
\item There are times when we would like to speak of the limit which f approaches at a, even though there is no $\delta >$ 0 such that $f(x)$ is defined for
x satisfying
\begin{equation}
0 < |x - a| < \delta.
\end{equation}
\item For example, we want to distinguish the behavior of the two functions shown in Figure 16, even though they are not defined for numbers less than a.

\item For the function of Figure 16(a) we write
\begin{equation}
\lim_{x \to a^{+}}{f(x)=l}\text{ or }\lim_{x \downarrow a^{-}}{f(x)=l}.
\end{equation}
\begin{center}
\includegraphics{figure/chapter5/16.png}
\end{center}
\item (The symbols on the left are read: the limit of $f(x)$ as x approaches
a from above.)
\item These ``limits from above" are obviously closely related to ordinary limits, and the definition is very similar:
\bull{ $\displaystyle\lim_{x \to a^{+}}{f(x)=l}$ means that for every
$\varepsilon>$ 0 there is a $\delta>$ 0 such that, for all x, if }
\begin{equation}
0<x-a<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon.\end{equation}
\item (The condition `` $0< x-a <\delta$" is equivalent to ``$ 0<|x-a|<\delta$ and $x > a$".)
\item ``Limits from below" (Figure 17) are defined similarly:
\begin{itemize}
\item $\displaystyle\lim_{x \to a^{+}}{f(x)=l}$
\item (or $\displaystyle\lim_{x \uparrow a^{+}}{f(x)=l}$) means that for every
$\varepsilon >$ 0 there is a $\delta>$ 0 such that, for all x, if
\end{itemize}
\begin{equation}
0<a-x<\delta,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon.
\end{equation}
\begin{center}
\includegraphics{figure/chapter5/17.png}

```
\end{center}
\item It is quite possible to consider limits from above and below even if $f$ is
defined for numbers both greater and less than $a$.
\begin{center}
\includegraphics[scale=0.7]{figure/chapter5/13.jpg}\\
\end{center}
\item Thus, for the function $f$ of Figure 13, we have
\begin{equation}\lim_{x \to 0^{+}}{f(x)}=1\text{ and }\displaystyle\lim_{x \to
0^{-}}{f(x)}=-1.\end{equation}
\item It is an easy exercise (Problem 29) to show that \bull{$\displaystyle\lim_{x \to
a}{f(x)}$ exists if and only if $\displaystyle\lim_{x \to a^{+}}{f(x)}$ and
$\displaystyle\lim_{x \to a^{-}}{f(x)}$ both exist and are equal.}
\item Like the definitions of limits from above and below, which have been smuggled
into the text informally, there are other modifications of the limit concept which will
be found useful.
\item In Chapter 4 it was claimed that if $x$ is large, then $\sin \frac{1}{x}$ is close
to 0.
\item This assertion is usually written
\begin{equation}\displaystyle\lim_{x\to \infty} \sin \frac{1}{x} = 0 \end{equation}
\begin{center}
\includegraphics{figure/chapter5/18.png}
\end{center}
\item The symbol $\displaystyle\lim_{x\to \infty}{f(x)}$ is read ``the limit of
$f(x)$ as $x$ approaches $\infty$ ," or ``as $x$ becomes infinite," and a limit of the
form $\displaystyle\lim_{x\to \infty}{f(x)}$ is often called a limit at infinity.
\item Figure 18 illustrates a general situation where $\displaystyle\lim_{x\to
\infty}{f(x)} =l $ .
\item Formally, $\displaystyle\lim_{x\to \infty}{f(x)} = l$ means that for every
$\varepsilon > 0$ there is a number $N$ such that, for all $x$, if
\begin{equation}
x>N,
\end{equation}
then
\begin{equation}
|f(x)-l|<\varepsilon.
\end{equation}

\item The analogy with the definition of ordinary limits should be clear:
\begin{itemize}
\item whereas the condition ``$0<|x-a|<\delta$" expresses the fact that $x$ is close to
$a$,
\item the condition "$x > N$" expresses the fact that $x$ is large.
\end{itemize}
\item We have spent so little time on limits from above and below, and at infinity,
\begin{itemize}
\item because the general philosophy behind the definitions should be clear if you
understand the definition of ordinary limits
\item (which are by far the most important).
\end{itemize}
```

\item Many exercises on these definitions are provided in the Problems, which also contain several other types of limits which are occasionally useful.

\end{enumerate}
\subsection{Problem}
\begin{enumerate}
\item Find the following limits.
\begin{itemize}
\item (These limits all follow, after some algebraic manipulations, from the various parts of \\ Theorem 2;
\item be sure you know which ones are used in each case, but don't bother listing them.)
\end{itemize}
\begin{enumerate}
\item $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x+1}.$
\item $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x-2}.$
\item $\displaystyle\lim_{x\to 3}\frac{x^3-8}{x-2}.$
\item $\displaystyle\lim_{x\to y}\frac{x^n-y^n}{x-y}.$
\item $\displaystyle\lim_{y\to x}\frac{x^n-y^n}{x-y}.$
\item $\displaystyle\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}.$
\end{enumerate}
\item Find the following limits.
\begin{enumerate}
\item $\displaystyle\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}.$
\item $\displaystyle\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x}.$
\item $\displaystyle\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}.$
\end{enumerate}
\item In each of the following cases, find a δ such that $|f(x)-l| <\varepsilon $ for all x satisfying $0 < |x -a| <\delta$.
\begin{enumerate}
\itemsep1em
\item $f(x) = x^4;l=a^4.$
\item $f(x) = \frac{1}{x};a=1, l=1.$
\item $f(x) = x^4+\frac{1}{x};a=1, l=2.$
\item $f(x) = \frac{x}{1+sin^2{x}};a=0, l=0.$
\item $f(x) = \sqrt{|x|}; a=0, l=0.$
\item $f(x) = \sqrt{x};a=1, l=1.$
\end{enumerate}
\item For each of the functions in Problem 4-17, decide for which numbers a the limit $\displaystyle\lim_{x\to a}{f(x)}$ exists.
\item
\begin{enumerate}
\item Do the same for each of the functions in Problem 4-19.
\item Same problem, if we use infinite decimals ending in a string of O's instead of those ending in a string of 9's.
\end{enumerate}

\item Suppose the functions f and g have the following property: for all $\varepsilon>0$ and all x,

\begin{equation}\text{if }0<|x-
2|<\sin^2{\Bigg(\frac{\varepsilon^2}{9}\Bigg)+\varepsilon},\text{ then }|f(x)-
2|<\varepsilon\end{equation}
\begin{equation}\text{if }0<|x-2|<\varepsilon^2,\text{ then }|g(x)-
4|<\varepsilon\end{equation}
\bull{For each $\varepsilon>0$ find a $\delta>0$ such that, for all x,}
\begin{enumerate}
\item if $0<|x-2|<\delta$, then $|f(x)+g(x)-6|<\varepsilon$
\item if $0<|x-2|<\delta$, then $|f(x)g(x)-8|<\varepsilon$
\item if $0<|x-2|<\delta$, then $\left|\frac{1}{g(x)}-\frac{1}{4}\right|<\varepsilon$
\item if $0<|x-2|<\delta$, then $\left|\frac{f(x)}{g(x)}-\frac{1}{2}\right|<\varepsilon$
\end{enumerate}
\item Give an example of a function f for which the following assertion is $false$:
\bull{
If $|f(x)-l|<\varepsilon$ when $0<|x-a|<\delta$, then $|f(x)-
l|<\frac{\varepsilon}{2}$ when $0<|x-a|<\frac{\delta}{2}.$}
\item
\begin{enumerate}
\item If $\displaystyle\lim_{x\to a}{f(x)}$ and $\displaystyle\lim_{x\to a}{g(x)}$ do
not exist, can $\displaystyle\lim_{x\to a}{[f(x)+g(x)]}$ or $\displaystyle\lim_{x\to
a}{f(x)\cdot g(x)}$ exist?
\item If $\displaystyle\lim_{x\to a}{f(x)}$ exist and $\displaystyle\lim_{x\to
a}{[f(x)+g(x)]}$ exist, must $\displaystyle\lim_{x\to a}{g(x)}$ exist?
\item If $\displaystyle\lim_{x\to a}{f(x)}$ exist and $\displaystyle\lim_{x\to
a}{g(x)}$ does not exist, can $\displaystyle\lim_{x\to a}{[f(x)+g(x)]}$ exist?
\item If $\displaystyle\lim_{x\to a}{f(x)}$ exist and $\displaystyle\lim_{x\to
a}{f(x)g(x)}$ exist, does it follow that $\displaystyle\lim_{x\to a}{g(x)}$ exists?
\end{enumerate}
\item Prove that $\displaystyle\lim_{x\to a}{f(x)} = \displaystyle\lim_{h\to
0}{f(a+h)}$.

\begin{itemize}
\item (This is mainly an exercise in understanding what the terms mean.)
\end{itemize}
\item
\begin{enumerate}
\item Prove that $\displaystyle\lim_{x\to a}{f(x)} = l$ if and only if
$\displaystyle\lim_{x\to a}{[f(x)-l]=0}$.
\begin{itemize}
\item First see why the assertion is obvious; then provide a rigorous proof.
\item In this chapter most problems which ask for proofs should be treated in the same
way.)
\end{itemize}
\item Prove that $\displaystyle\lim_{x\to 0}{f(x)} = \displaystyle\lim_{x\to a}{f(x-
a)}$.
\item Prove that $\displaystyle\lim_{x\to 0}{f(x)} = \displaystyle\lim_{x\to
0}{f(x^3)}$.
\item Give an example where $\displaystyle\lim_{x\to 0}{f(x^2)}$ exists, but
$\displaystyle\lim_{x\to 0}{f(x)}$ does not.
\end{enumerate}

\item
\begin{itemize}
\item
Suppose there is a $\delta>0$ such that $f(x) = g(x)$ when $0 < |x-a| <\delta$.
\item Prove that $\displaystyle\lim_{x\to a}{f(x)} = \displaystyle\lim_{x\to a}{g(x)}$.
\item In other words, $\displaystyle\lim_{x\to a}{f(x)}$ depends only on the values of $f(x)$ for x near a.
\item This fact is often expressed by saying that limits are a ``local property''.
\item (It will clearly help to use $\delta^{'}$, or some other letter, instead of δ, in the definition of limits.
\end{itemize}
\item
\begin{enumerate}
\item Suppose that $f(x) = g(x)$ for all x.\bull{ Prove that $\displaystyle\lim_{x\to a}{f(x)} \leq \displaystyle\lim_{x\to a}{g(x)}$, provided that these limits exist.}
\item How can the hypotheses by weakened?
\item If $f(x) < g(x)$ for all x, does it necessarily follow that $\displaystyle\lim_{x\to a}{f(x)} < \displaystyle\lim_{x\to a}{g(x)}$
\end{enumerate}
\item Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\displaystyle\lim_{x\to a}{f(x)} = \displaystyle\lim_{x\to a}{h(x)}$.
\bull{ Prove that $\displaystyle\lim_{x\to a}{f(x)} = \displaystyle\lim_{x\to a}{h(x)}$, and exist $\displaystyle\lim_{x\to a}{f(x)} = \displaystyle\lim_{x\to a}{g(x)} = \displaystyle\lim_{x\to a}{h(x)}$ (Draw a picture!)}
\item
\begin{enumerate}
\item Prove that if and $\displaystyle\lim_{x\to a}{\frac{f(x)}{x}} = l $ and $b \neq 0$,then $\displaystyle\lim_{x\to a}{\frac{f(bx)}{x}} = bl$. \bull{Hint: write $\frac{f(x)}{x}=b[\frac{f(bx)}{bx}]$.}
\item What happens if b = 0?
\item Part (a) enable us to find $\displaystyle\lim_{x\to 0}{\frac{\sin(2x)}{x}}$ in terms of $\displaystyle\lim_{x \to a}{\frac{\sin(x)}{x}}$.
\item Find this limit in another way.
\end{enumerate}
\item Evaluate the following limits in terms of the number $\alpha = \displaystyle\lim_{x \to 0}{\frac{\sin(x)}{x}}$
\begin{enumerate}
\item $\displaystyle\lim_{x \to 0}{\frac{\sin(2x)}{x}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{\sin(ax)}{\sin(bx)}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{\sin^2(2x)}{x}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{\sin^2(2x)}{x^2}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{1-\cos(x)}{x^2}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{\tan^2(x)+2x}{x+x^2}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{x\sin(x)}{1-\cos(x)}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{\sin(x+h)-\sin(x)}{h}}$.
\item $\displaystyle\lim_{x \to 1}{\frac{\sin(x^2-1)}{x-1}}$.
\item $\displaystyle\lim_{x \to 0}{\frac{x^2 (3+\sin(x))}{1-\cos(x)}}$.
\item $\displaystyle\lim_{x \to 1}{(x^2-1)^3\sin(\frac{1}{x-1})^3}$.
\end{enumerate}

\item
\begin{enumerate}
\item Prove that if $\displaystyle\lim_{x \to a }{f(x)}=l$, then $\displaystyle\lim_{x \to a }{|f|(x)}=|l|$.
\item Prove that if $\displaystyle\lim_{x \to a }{f(x)}=l$ and $\displaystyle\lim_{x \to a }{g(x)}=m$, then $\displaystyle\lim_{x \to a }{\max(f,g)(x)}=\max(l,m)$ and similarly for min.
\end{enumerate}
\item
\begin{enumerate}
\item Prove that $\dislim_{x \to 0}{\frac{1}{x}}$ does not exist, i.e., show that $\dislim_{x \to 0}{\frac{1}{x}}=l$ is false for every number l.
\item Prove that $\dislim_{x \to 1}{\frac{1}{(x-1)}}$ does not exist.
\end{enumerate}
\item
\begin{itemize}
\item Prove that if $\dislim_{x \to a }{f(x)}=l$, then there is a number $\delta > 0$ and a number M such that $|f(x)|<M$ if $0<|x-a|<\delta$.
\item What does this mean pictorially?
\item Hint: Why does it suffice to prove that $l-1<f(x)<l+1$ for $0 < |x-a| <\delta$?
\end{itemize}
\item Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x, then $\dislim_{x \to a}{f(x)}$ does not exist for any a.
\item Prove that if $f(x) = x$ for rational x, and $f(x) = -x$ for irrational x, then $\dislim_{x \to a}{f(x)}$ does not exist if $a \neq 0$.

\item
\begin{enumerate}
\item Prove that if $\dislim_{x \to 0}{g(x)}=0$, then $\dislim_{x \to 0}{g(x)}\sin\left(\frac{1}{x}\right) = 0 $.
\item
\begin{itemize}
\item Generalize this fact as follows: If $\dislim_{x \to 0}{g(x)}=0$ and $|h(x)| \leq M$ for all x, \\then $\dislim_{x \to 0}{g(x)h(x)} = 0$.
\item (Naturally it is unnecessary to do part (a) if you succeed in doing part (b);)
\item actually the statement of part (b) may make it easier than (a)--that's one of the values of generalization.
\end{itemize}
\end{enumerate}
\item
\begin{itemize}
\item Consider a function f with the following property: if g is any function for which $\dislim_{x \to 0}{g(x)}$ does not exist, then $\dislim_{x \to 0}{[g(x)+f(x)]}$ also does not exist.
\item Prove that this happens if and only if $\dislim_{x \to 0}{f(x)}$ does exist.
\item Hint: This is actually very easy: the assumption that $\dislim_{x \to 0}{f(x)}$ does not exist leads to an immediate contradiction if you consider the right g.
\end{itemize}

\item
\begin{itemize}
\item This problem is the analogue of Problem 22 when $f +g$ is replaced by $f \cdot g$.
\item In this case the situation is considerably more complex, and the analysis requires several steps (those in search of an especially challenging problem can attempt an independent solution).

\item Suppose that $\dislim_{x \to 0}{f(x)}$ exists and is $\neq 0$.Prove that if $\dislim_{x \to 0}{g(x)}$ does not exist, then $\dislim_{x \to 0}{f(x)g(x)}$ also does not exist.
\item Prove the same result if $\dislim_{x \to 0}{|f(x)|} = \infty$.(The precise definition of this sort of limit is given in Problem 37.
\item
\begin{itemize}
\item Prove that if neither of these two conditions holds, then there is a function g such that $\dislim_{x \to 0}{g(x)}$ does not exist, but $\dislim_{x \to 0}{ f(x)g(x)}$ does exist.
\item Hint: Consider separately the following two cases:
\begin{enumerate}[1.]
\item for some $\varepsilon>0$ we have $|f(x)|<\varepsilon$ for all sufficiently small x.
\item For every $\varepsilon > 0$, there are arbitrarily small x with $|f(x)| < \varepsilon$. In the second case, begin by choosing points x_n with $|x_n| < \frac{1}{n}$ and $|f(x_n) < \frac{1}{n}|$.
\end{enumerate}
\end{itemize}

\end{itemize}
\item Suppose that A_n is, for each natural number n, some finite set of numbers in [0, 1], and that A_n and A_m have no members in common if $m \neq n$.
\bull{ Define f as follow: }
\begin{equation}
f(x) = \begin{cases}
\frac{1}{n}, x \text{ in } A_n \\
0, x \text{ not in } A_n \text{ for any } n. \\
\end{cases}
\end{equation}
prove that $\dislim_{x \to a}{f(x)} = 0$ for all a in [0,1].
\item Explain why the following definitions of $\dislim_{x \to a}{f(x)} = l$ are all correct:
\bull{ For every $\delta > 0$ there is an $\varepsilon > 0$ such that, for all x,}
\begin{enumerate}
\item if $0<|x-a|<\varepsilon$, then $|f(x)-l|<\delta$.
\item if $0<|x-a|<\varepsilon$, then $|f(x)-l|\leq\delta$.
\item if $0<|x-a|<\varepsilon$, then $|f(x)-l|<5\delta$.
\item if $0<|x-a|<\frac{\varepsilon}{10}$, then $|f(x)-l|<\delta$.
\end{enumerate}
\item Give examples to show that the following definitions of $\dislim_{x \to a}{f(x)=l$ are not correct.

\begin{enumerate}
\item For all $\delta>0$ there is an $\varepsilon>0$ such that if $0 < |x-a| < \delta$, then $|f(x)-l|<\varepsilon$.
\item For all $\varepsilon>0$ there is a $\delta>0$ such that if $|f(x)-l|<\varepsilon$, then $0 < |x-a| < \delta$.
\end{enumerate}
\item For each of the functions in Problem 4-17 indicate for which numbers a the one-sided limits $\dislim_{x \to a^+}{f(x)}$ and $\dislim_{x \to a^-}{f(x)}$ exist.
\item
\begin{enumerate}
\item Do the same for each of the functions in Problem 4-19.
\item Also consider what happens if decimals ending in O's are used instead of decimals ending in 9's.
\end{enumerate}
\item Prove that $\dislim_{x \to a}{f(x)}$ exist if $\dislim_{x \to a^+}{f(x)} = \dislim_{x \to a^-}{f(x)}$

\item Prove that
\begin{enumerate}
\item $\dislim_{x \to 0^+}{f(x)} = \dislim_{x \to 0^-}{f(-x)}$.
\item $\dislim_{x \to 0}{f(|x|)} = \dislim_{x \to 0^+}{f(x)}$.
\item $\dislim_{x \to 0}{f(x^2)} = \dislim_{x \to 0^+}{f(x)}$.
\end{enumerate}
\begin{itemize}
\item (These equations, and others like them, are open to several interpretations.
\item They might mean only that the two limits are equal if they both exist;
\item or that if a certain one of the limits exists, the other also exists and is equal to it;
\item or that if either limit exists, then the other exists and is equal to it.
\item Decide for yourself which interpretations are suitable.)
\end{itemize}
\item
\begin{itemize}
\item Suppose that $\dislim_{x \to a^-}{f(x)}<\dislim_{x \to a^+}{f(x)}$.
\item (Draw a picture to illustrate this assertion.)
\item Prove that there is some $\delta>0$ such that $f(x) < f(y)$ whenever $x <a < y$ and $|x-a|<\delta$ and $|y-a|<\delta$.
\item Is the converse true?
\end{itemize}
\item
\begin{itemize}
\item Prove that $\dislim_{x \to \infty}{\frac{(a_nx^n+\cdots+a_0)}{(b_mx^m+\cdots+a_0)}}$ exists if and only if $m \geq n$.
\item What is the limit when $m = n$? When $m > n$?
\item Hint: the one easy limit is $\dislim_{x \to \infty}{\frac{1}{x^k}} =0$;
\item do some algebra so that this is the only information you need.
\end{itemize}

\item Find the following limits.
\begin{enumerate}

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\item $\dislim_{x \to \infty}{\frac{x+\sin^3(x)}{5x+6}}$.
\item $\dislim_{x \to \infty}{\frac{x+\sin(x)}{x^2+5}}$.
\item $\dislim_{x \to \infty}{\sqrt{x^2+x}-x}$.
\item $\dislim_{x \to \infty}{\frac{x^2(1+\sin^2(x))}{(x+\sin(x))^2}}$.
\end{enumerate}
\item Prove that $\dislim_{x \to 0^+}{f(\frac{1}{x})} = \dislim_{x \to \infty}{f(x)}$
\item Find the following limits in terms of the number $\alpha = \dislim_{x \to
0}{\frac{\sin(x)}{x}}$
\begin{enumerate}
\item $\dislim_{x \to \infty}{\frac{\sin(x)}{x}}$.
\item $\dislim_{x \to \infty}{x\sin(\frac{1}{x})}$.
\end{enumerate}
\item Define ``$\dislim_{x \to -\infty}{f(x)}=l$."
\begin{enumerate}
\item Find $\dislim_{x \to -
\infty}{\frac{(a_nx^n+\cdots+a_0)}{(b_mx^m+\cdots+a_0)}}$
\item Prove that $\dislim_{x \to \infty}{f(x)} = \dislim_{x \to -\infty}{f(-x)}$.
\item Prove that $\dislim_{x \to 0^-}{f(\frac{1}{x})} = \dislim_{x \to -\infty}{f(x)}$.
\end{enumerate}
\item
\begin{itemize}
\item We define $\dislim_{x \to a}{f(x)} = \infty $ to mean that for all N
\item there is a $\delta>0$ such that, for all $x$, if $0 < |x-a| < \delta$, then $f(x) >
N$.
\item (Draw an appropriate picture!)
\end{itemize}
\begin{enumerate}
\item Show that $\dislim_{x \to 3}{\frac{1}{(x-3)^2}}=\infty$.
\item Prove that if $f(x)>\varepsilon>0$ for all $x$, and $\dislim_{x \to a}{g(x)} =
0$, then
\begin{equation}\dislim_{x \to a}{\frac{f(x)}{|g(x)|}} = \infty.\end{equation}
\end{enumerate}
\item
\begin{enumerate}
\item Define $\dislim_{x \to a^+}{f(x)}=\infty$,$\dislim_{x \to a^-}{f(x)}=\infty$,
and $\dislim_{x \to a}{f(x)}=\infty$.\bull{ (Or at least convince yourself that you
could write down the definitions if you had the energy.\item How many other such
symbols can you define?)}
\item Prove that $\dislim_{x \to 0^+}{\frac{1}{x}} = \infty$.
\item Prove that $\dislim_{x \to 0^+}{f(x)}=\infty$ if and only if $\dislim_{x \to
\infty}{f(\frac{1}{x})} = \infty$.
\end{enumerate}

\item Find the following limits, when they exist.
\begin{enumerate}
\item $\dislim_{x \to \infty}{\frac{x^3+4x-7}{7x^2-x+1}}$
\item $\dislim_{x \to \infty}{x(1+\sin^2(x))}$
\item $\dislim_{x \to \infty}{x\sin^2(x)}$
\item $\dislim_{x \to \infty}{x^2\sin(\frac{1}{x})}$
\item $\dislim_{x \to \infty}{\sqrt{x^2+2x}-x}$
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\item $\dislim_{x \to \infty}{x(\sqrt{x+2}-\sqrt{x})}$
\item $\dislim_{x \to \infty}{\frac{\sqrt{|x|}}{x}}$
\end{enumerate}
\item
\begin{enumerate}
\item Find the perimeter of a regular n-gon inscribed in a circle of radius r; use radian measure for any trigonometric functions involved.[Answer: $2rn \sin(\frac{\pi}{n})$.]
\item What value does this perimeter approach as n becomes very large?
\end{enumerate}
\item
\begin{itemize}
\item After sending the manuscript for the first edition of this book off to the printer,
\item I thought of a much simpler way to prove that $\dislim_{x \to a}{x^2} = a^2$ and $\dislim_{x \to a}{x^3} = a^3$, without going through all the factoring tricks on page 95.
\item Suppose, for example, that we want to prove that $\dislim_{x \to a}{x^2} = a^2$, where $a > 0$.
\item Given $\varepsilon>0$, we simply let δ be the minimum of
\begin{equation}
\sqrt{a^2+\varepsilon}-a
\end{equation}
and
\begin{equation}
a - \sqrt{a^2-\varepsilon}
\end{equation}
(see Figure 19); then
\begin{equation}
|x-a|<\delta
\end{equation}
implies that
\begin{equation}
\sqrt{a^2-\varepsilon} < x < \sqrt{a^2+\varepsilon},
\end{equation}
so
\begin{equation}
a^2-\varepsilon < x^2 < a^2+\varepsilon,
\end{equation}
or
\begin{equation}
|x^2-a^2|<\varepsilon.
\end{equation}
\item It is fortunate that these pages had already been set, so that I couldn't make these changes, because this `` Proof ” is completely fallacious.
\item Wherein lies the fallacy?

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\end{itemize}
\end{enumerate}
\newpage
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\section{Continuous Function (p.113)}
\subsection{Context}
\begin{enumerate}
\item If $f$ is an arbitrary function, it is not necessarily true that
\begin{equation}\dislim_{x \to a}{f(x)} = f(a)\end{equation}
\item In fact, there are many ways this can fail to be true.
\item For example, $f$ might not even be defined at $a$, in which case the equation makes no sense (Figure 1).
\begin{center}
\includegraphics[scale=0.4]{figure/chapter6/1.jpg}\\
\end{center}
\item Again, $\dislim_{x \to a}{f(x)}$ might not exist (Figure 2).
\begin{center}
\includegraphics[scale=0.4]{figure/chapter6/2.jpg}\\
\end{center}

\item Finally, as illustrated in Figure 3, even if $f$ is defined at $a$ and $\dislim_{x \to a}{f(x)}$ exists, the limit might not equal $f(a)$.
\begin{center}
\includegraphics[scale=0.6]{figure/chapter6/3.jpg}\\
\end{center}
\item We would like to regard all behavior of this type as abnormal and honor, with some complimentary designation, functions which do not exhibit such peculiarities.
\item The term which has been adopted is ``continuous".
\item Intuitively, a function $f$ is continuous if the graph contains no breaks, jumps, or wild oscillations.
\item Although this description will usually enable you to decide whether a function is continuous simply by looking at its graph (a skill well worth cultivating)
\bull{ it is easy to be fooled, and the precise definition is \textit{very} important.}


\begin{framed}
DEFINITION \\
The function $f$ is continuous at $a$ if
\begin{equation}\dislim_{x \to a}{f(x)} = f(a).\end{equation}
\end{framed}
\item We will have no difficulty finding many examples of functions which are, or are not, continuous at some number $a$
\bull{--- every example involving limits provides an example about continuity, and Chapter 5 certainly provides enough of these.}
\item The function $f(x) = \sin\left(\frac{1}{x}\right)$ is not continuous at 0, because it is not even defined at 0, and the same is true of the function $g(x) = x \sin\left(\frac{1}{x}\right)$.
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\item On the other hand, if we are willing to extend the second of these functions, that
is, if we wish to define a new function $G$ by
\begin{equation}
\begin{aligned}
G(x) = \begin{cases}
x\sin\left(\frac{1}{x}\right) &x\neq0\\
a&x = 0,\\
\end{cases}
\end{aligned}
\end{equation}
\item then the choice of $a = G(0)$ can be made in such a way that $G$ will be
continuous at 0
\bull{--- to do this we can (if fact, we must) define $G(0)= 0$ (Figure 4).}
\begin{center}
\includegraphics[scale=0.4]{figure/chapter6/4.jpg}\\
\end{center}
\item This sort of extension is not possible for $f$;
\bull{ if we define
\begin{equation}
\begin{aligned}
f(x) = \begin{cases}
\sin\left(\frac{1}{x}\right) & x\neq0\\
a&x = 0,\\
\end{cases}
\end{aligned}
\end{equation}
then $f$ will not be continuous at 0, no matter what $a$ is, because $\dislim_{x \to
a}{f(x)}$ does not exist.}
\item The function
\begin{equation}
f(x) = \begin{cases}
x &\text{x rational}\\
0 &\text{x irrational,}\\
\end{cases}
\end{equation}
is not continuous at $a$, if $a\neq 0$, since $\dislim_{x \to a}{f(x)}$ does not exist.
\item However, $\dislim_{x \to a}{f(x)} = 0 = f(0)$, so $f$ is continuous at precisely
one point 0.
\item The functions $f(x) = c$, $g(x) = x$, and $h(x) = x^2$ are continuous at all
numbers $a$, since
\begin{equation} \lim_{x \to a} {f(x)} = \lim_{x \to a} {c} = c = f(a), \end{equation}
\begin{equation} \lim_{x \to a} {g(x)} = \lim_{x \to a} {x} = a = g(a),
\end{equation}
\begin{equation} \lim_{x \to a} {h(x)} = \lim_{x \to a} {x^2} = a^2 = h(a),
\end{equation}
\item Finally, consider the function
\begin{equation}
f(x) = \begin{cases}
0 &\text{x irrational}\\
\dfrac{1}{q} &x=\dfrac{p}{q} \text{ in lower terms.}\\
```

\end{cases}
\end{equation}
\item In Chapter 5 we showed that $\dislim_{x \to a}{f(x)} =0$ for all a.
\item Since $0 = f(a)$ only when a is irrational, this function is continuous at a if a is irrational, but not if a is rational.
\item It is even easier to give examples of continuity if we prove two simple theorems.

\begin{framed}
THEOREM 1\\
If f and g are continuous at a, then
\begin{enumerate}[1.]
\item $f+g$ is continuous at a.
\item $f \cdot g$ is continuous at a.
\item Moreover, if $g(a) \neq 0$, then $\frac{1}{g}$ is continuous at a.

\end{enumerate}
\end{framed}
Proof:\\
\item Since f and g are continuous at a,
\begin{equation}\dislim_{x \to a}{f(x)} = f(a) \text{ and } \dislim_{x \to a}{g(x)} = g(a)\end{equation}
\item By Theorem 2(1) of Chapter 5 this implies that
\begin{equation}\dislim_{x \to a}{f(x)+g(x)} = f(a)+g(a) = (f+g)(a),\end{equation}
which is just the assertion that $f+g$ is continuous at a.
\item The proofs of parts (2) and (3) are left to you.$\blacksquare$

\item Starting with the functions $f(x) = c$ and $f(x) = x$, which are continuous at a, for every a, we can use Theorem 1 to conclude that a function
\begin{equation}f(x) = \dfrac{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_0}{c_mx^m+c_{m-1}x^{m-1}+\cdots+c_0}\end{equation}
is continuous at every point in its domain.
\item But it is harder to get much further than that.
\item When we discuss the sine function in detail it will be easy to prove that $\sin$ is continuous at a for all a; let us assume this fact meanwhile.
\item A function like
\begin{equation}f(x) = \dfrac{\sin^2(x)+x^2+x^4\sin(x)}{\sin^{27}(x)+4x^2\sin^2(x)}\end{equation}
can now be proved continuous at every point in its domain.
\item But we are still unable to prove the continuity of a function like $f(x)=\sin(x^2)$;
\bull{ we obviously need a theorem about the composition of continuous functions.}
\item Before stating this theorem, the following point about the composition of continuous functions.
\item If we translate the equation $\dislim_{x \to a }{f(x)} = f(a)$ according to the definition of limits, we obtain for every $\varepsilon>0$ there is $\delta>0$ such that, for all x, if
\begin{equation}
0<|x-a|<\delta,
\end{equation}
then

```
\begin{equation}
|f(x)-f(a)|<\varepsilon.
\end{equation}
\item But in this case, where the limit is $f(a)$, the phrase
\begin{equation}0<|x-a|<\delta\end{equation}
may be change to the simpler condition
\begin{equation}|x-a|<\delta.\end{equation}
\item Since if $x=a$ it is certainly true that
\begin{equation}
|f(x)-f(a)|<\varepsilon.
\end{equation}
\begin{framed}
THEOREM 2\\
If $g$ is continuous at $a$, and $f$ is continuous at $g(a)$, then $f \circ g $ is
continuous at $a$. (Notice that $f$ is required to be continuous at $g(a)$, not at $a$.)
\end{framed}

\begin{proof}\ \\
\begin{enumerate}[a.]
\item Let $\varepsilon > 0$.
\item We wish to find a $\delta > 0$ such that for all $x$.
\item If $|x-a|<\delta$, then $| (f \circ g)(x)-(f \circ g)(a) |<\varepsilon$.
\item i.e., $|f(g(x))-f(g(a))|<\varepsilon$
\end{enumerate}
\end{proof}

\item We first use continuity of $f$ to estimate how close $g(x)$ must be to $g(a)$ in
order for this inequality to hold.
\item Since $f$ is continuous at $g(a)$, there is a $\delta ' > 0$ such that for all $y$, if
\begin{equation}
|y-g(a)|<\delta',
\end{equation}
then
\begin{equation}
|f(y)-f(g(a))|<\varepsilon
\end{equation}
\item In particular, this mean that if
\begin{equation}
|g(x)-g(a)|<\delta',
\end{equation}
then
\begin{equation}
|f(g(x))-f(g(a))|<\varepsilon.
\end{equation}
\item We now use continuity of $g$ to estimate how close $x$ must be to $a$ in order
for the inequality
\begin{equation}
|g(x) - g(a)| < \delta
\end{equation}
to hold.
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\item The number δ is a positive number just like any other positive number;
\bull{we can therefore take δ' as the ε(!) in the definition of continuity of g at a.}
\item We conclude that there is a $\delta >0$ such that, for all x, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|g(x)-g(a)|<\delta'.
\end{equation}
\item Combining (2.19) and (2.20) we see that for all x, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(g(x))-f(g(a))|< \varepsilon
\end{equation}
\item We can now reconsider the function
\begin{equation}
f(x) = \begin{cases}
\sin\left(\frac{1}{x}\right) &x\neq0\\
0&x = 0,\\
\end{cases}
\end{equation}
\item We have already noted that f is continuous at 0.
\item A few applications of Theorems 1 and 2, together with the continuity of $\sin$, show that f is also continuous at a, for $a \neq 0$.
\item Functions like
\begin{equation}
f(x) = \sin(x^2+\sin(x+\sin^2(x^3)))
\end{equation}
should be equally easy for you to analyze.
\item
\begin{itemize}
\item The few theorems of this chapter have all been related to continuity of functions at a single point,
\item but the concept of continuity doesn't begin to be really interesting
\item until we focus our attention on functions which are continuous at all points of some interval.
\end{itemize}
\item If f is continuous at x for all x in (a, b), then f is called continuous on (a, b).
\item Continuity on a closed interval must be defined a little differently; a function f is called continuous on $[a,b]$ if

\begin{enumerate}[1.]
\item f is continuous at x for all x in (a,b),
\item $\dislim_{x \to a^+}{f(x)} = f(a)$ and $\dislim_{x \to b^-}{f(x)} = f(b).$

\end{enumerate}

\item Functions which are continuous on an interval are usually regarded as especially well behaved;
\bull{ indeed continuity might be specified as the first condition which a ``reasonable" function ought to satisfy.}
\item A continuous function is sometimes described, intuitively, as one whose graph can be drawn without lifting your pencil from the paper.
\item Consideration of the function
\begin{equation}
f(x) = \begin{cases}
\sin\left(\frac{1}{x}\right) &x\neq0\\
0&x = 0,\\
\end{cases}
\end{equation}
shows that this description is a little too optimistic,
\bull{ but it is nevertheless true that there are many important results involving functions which are continuous on an interval.}
\item There theorems are generally much harder than the ones in this chapter, but there is a simple theorem which forms a bridge between the two kinds of results.
\item The hypothesis of this theorem requires continuity at only a single point, but the conclusion describes the behavior of the function on some interval containing the point.
\item Although this theorem is really a lemma for later arguments, it is included here as a preview of things to come.

\begin{framed}
THEOREM 3
\begin{enumerate}[1.]
\item Suppose f is continuous at a, and $f(a) > 0$.
\item Then there is a number $\delta >0$ such that $f(x) > 0$ for all x satisfying $|x-a| < \delta$.
\item Similarly, if $f(a) <0$, then there is a number $\delta >0$ such that $f(x) < 0$ for all x satisfying $|x-a| <\delta$.
\end{enumerate}
\end{framed}

\begin{proof}
\begin{enumerate}[1.]
\item Consider the case $f(a) > 0.$
\item Since f is continuous at a, if $\varepsilon > 0$ there is a $\delta > 0 $ such that, for all x, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-f(a)|<\varepsilon.
\end{equation}

```
\item Since $f(a) > 0$ we can take $f(a)$ as the $\varepsilon$.
\item Thus there is $\delta >0$ so that for all $x$, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-f(a)|<f(a),
\end{equation}
and this last inequality implies $f(x)>0$.
\item A similar proof can be given in the case $f(a) < 0$;
\bull{ take $\varepsilon = -f(a)$.}
\item Or one can apply the first case to the function -$f$.
\end{enumerate}
\end{proof}
\end{enumerate}

\subsection{Problem}
\begin{enumerate}
\item For which of the following functions $f$ is there $a$ continuous function
$F$ with domain $\mathbb{R}$ such that $F(x)=f(x)$ for all $x$ in the domain of
$f$
\begin{enumerate}
\item $f(x) =\dfrac{x^2-4}{x-2}$
\item $f(x) =\dfrac{|x|}{x}$
\item $f(x) =0$, $x$ irrational.
\item $f(x)=\dfrac{1}{q},x=\dfrac{p}{q}$ rational in lowest terms.
\end{enumerate}
\item At which points are the functions of Problems 4-17 and 4-19 continuous?
\item
\begin{enumerate}
\item Suppose that $f$ is a function satisfying $|f(x)| \leq |x| $ for all $x$.
\begin{itemize}
\item Show that $f$ is continuous at 0. (Notice that $f(0)$ must equal 0.)
\end{itemize}
\item Give an example of such a function $f$ which is not continuous at any $a \neq
0$.
\item Suppose that $g$ is continuous at 0 and $g(0)$ = 0, and $|f(x)| \leq |g(x)|$.
\begin{itemize}
\item Prove that $f$ is continuous at 0.
\end{itemize}
\end{enumerate}

\item Give an example of a function $f$ such that $f$ is continuous nowhere, but
$|f|$ is continuous everywhere.
\item For each number $a$, find a function which is continuous at $a$, but not at any
other points.
\item
\begin{enumerate}
```

\item Find a function f which is discontinuous at 1, $\dfrac{1}{2}$,
$\dfrac{1}{3}$, $\dfrac{1}{4}$, $\cdots$ but continuous at all other points.
\item Find a function f which is discontinuous at 1, $\dfrac{1}{2}$,
$\dfrac{1}{3}$, $\dfrac{1}{4}$, $\cdots$, and at 0, but continuous at all other points.
\end{enumerate}
\item Suppose that f satisfies $f (x + y) = f(x) + f(y)$, and that f is continuous
at 0.
\begin{itemize}
\item Prove that f is continuous at a for all a.
\end{itemize}
\item Suppose that f is continuous at a and $f(a) = 0$.
\begin{itemize}
\item Prove that if $a \neq 0$, then $f+\alpha$ is nonzero in some open interval
containing a.
\end{itemize}
\item
\begin{enumerate}
\item Suppose f is not continuous at a.
\begin{itemize}
\item Prove that for some number $\varepsilon > 0 $ there are numbers x arbitrarily
close to a with \\$f(x) - f(a) > \varepsilon$. \item Illustrate graphically.
\end{itemize}

\item Conclude that for some number $\varepsilon>0$ either there are numbers
x arbitrarily close to a with $f(x) < f(a) - \varepsilon $ or there are numbers
x arbitrarily close to a with $f(x) > f(a) + \varepsilon$.
\end{enumerate}
\item
\begin{enumerate}
\item Prove that if f is continuous at a, then so is $|f|$.
\item Prove that every continuous f can be written $f = E + O$, where E is even and
continuous and O is odd and continuous.
\item Prove that if f and g are continuous, then so are $\max(f,g)$ and
$\min(f,g)$
\item Prove that every continuous f can be written $f=g-h$, where g and h are
non-negative and continuous.
\end{enumerate}
\item Prove Theorem 1(3) by using Theorem 2 and continuity of the function
$f(x)=\dfrac{1}{x}.$
\item
\begin{enumerate}
\item Prove that f is continuous at l and $\dislim_{x \to a}{g(x)} = l$, then
$\dislim_{x \to a}{f(g(x))}=f(l)$.
\begin{itemize}
\item (You can go right back to the definitions,
\item but it is easier to consider the function G with $G(x)=g(x)$ for $x \neq a$,
and $G(a)=l$.)
\end{itemize}

\item Show that if continuity of f at l is not assumed, then it is not generally true that
\begin{equation}
\dislim_{x \to a}{f(g(x))} = f(\dislim_{x \to a}{g(x)}).
\end{equation}
\bull{ Hint: Try $f(x)=0$ for $x \neq l$, and $f(l)=1$.}
\end{enumerate}

\item
\begin{enumerate}
\item Prove that if f is continuous on $[a,b]$, then there is a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x)=f(x)$ for all x in $[a,b]$.
\bull{Hint: Since you obviously have a great deal of choice, try making g constant in $(-\infty,a]$ and $[b,\infty)$.}
\item Give an example to show that this assertion is false if $[a,b]$ is replaced by (a,b).
\end{enumerate}
\item
\begin{enumerate}
\item
\begin{itemize}
\item Suppose that g and h are continuous at a, and that $g(a)=h(a)$.
\item Define $f(x)$ to be $g(x)$ if $x \geq a$ and $h(x)$ if $x \leq a$.
\item Prove that f is continuous at a.
\end{itemize}

\item
\begin{itemize}
\item Suppose g is continuous on $[a,b]$ and h us continuous in $[b,c]$ and $g(b) = h(b)$.
\item Let $f(x)$ be $g(x)$ for x in $[a,b]$ and $h(x)$ for x in $[b,c]$.
\item Show that f is continuous on $[a,c]$.
\item (Thus, continuous functions can be ``pasted together"".)
\end{itemize}
\end{enumerate}

\item
\begin{enumerate}
\item
\begin{itemize}
\item Prove that following version of Theorem 3 for ``right-hand continuity"":
\item Suppose that $\dislim_{x \to a^+}{f(x)} =f(a)$, and $f(a)>0$. \item Then there is a number $\delta>0$ such that $f(x)>0$ for all x satisfying $0\leq x-a < \delta$.
\item Similarly, if $f(x)<0$ for all x satisfying $0 \leq x-a < \delta$.
\end{itemize}
\item Prove a version of Theorem 3 when $\dislim_{x \to b^-}{f(x)}=f(b)$.
\end{enumerate}
\item if $\dislim_{x \to a}{f(x)}$ exists, but is $\neq f(a)$, then f is said to have a \textbf{removable dis-continuity} at a.
\begin{enumerate}

```
\item
\begin{itemize}
\item If $f(x)= \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0 $ and $f(0)=1$, does
$f$ have a removable discontinuity at 0?
\item What if $f(x)=x\sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$, and $f(0)=1$?
\end{itemize}

\item
\begin{itemize}
\item Suppose $f$ has a removable discontinuity at $a$.
\item Let $g(x)=f(x)$ for $x \neq a$, and let $g(a)=\dislim_{x \to a }{f(x)}$.
\item Prove that $g$ is continuous at a.
\item (Don't work very hard; this is quite easy.)
\end{itemize}

\item
\begin{itemize}
\item Let $f(x)=0$ if x is irrational, and let
$f\left(\dfrac{p}{q}\right)=\dfrac{1}{q}$ if $\dfrac{p}{q}$ is in the lowest terms.
\item What is the function $g$ defined by $g(x) = \dislim_{y \to x}{f(y)}$?
\end{itemize}

\item
\begin{itemize}
\item Let $f$ be a function with the property that every point of discontinuity is a
removable discontinuity.
\item This means that $\dislim_{y \to x}{f(y))}$ exists for all $x$, but $f$ may be
discontinuous at some (even infinitely many) number $x$.
\item Define $g(x)= \dislim_{y \to x}{f(y)}$.
\item Prove that $g$ is continuous.
\item (This is not quite so easy as part(b).)
\end{itemize}

\item
\begin{itemize}
\item Is there a function which is discontinuous at every point, and which has only
removable discontinuities?
\item (It is worth thinking about this problem now, but mainly as a test of intuition;
\item even if you suspect the correct answer, you will almost certainly be unable to
prove it at the present time.
\item See Problem 22-33.)
\end{itemize}

\end{enumerate}
\item Now that we have discovered the fallacy, it is almost obvious what additional
property of the real numbers we need.
\item All we must do is say it properly and use it.
\item That is the business of the next chapter.
\end{enumerate}
\newpage
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\section{Three Hard Theorems (p.120)}
\subsection{Context}
\begin{enumerate}
\item This chapter is devoted to three theorems about continuous functions, and some
of their consequences.
\item The proofs of the three theorems themselves will not be given until the next
chapter, for reasons which are explained at the end of this chapter.

\begin{framed}
THEOREM 1 \\
If $f$ is continuous on $[a,b]$ and
\begin{equation}
f(a)<0<f(b),
\end{equation}
then there is some $x$ in $[a,b]$ such that \\
\begin{equation}
f(x)=0.
\end{equation}
\end{framed}

\item (Geometrically, this means that the graph of a continuous function which starts
below the horizontal axis and ends above it must cross this axis at some point, as in
Figure 1.)
\begin{center}
\includegraphics[scale=0.7]{figure/chapter7/1.jpg}\\
\end{center}


\begin{framed}
THEOREM 2 \\
If $f$ is continuous on $[a,b]$, then $f$ is bounded above on $[a,b]$, that is, there is
some number $N$ such that
\begin{equation}
f(x)<=N
\end{equation}
for all $x$ in $[a,b]$.
\end{framed}

\item (Geometrically, this theorem means that the graph of $f$ lies below some line
parallel to the horizontal axis, as in Figure 2.)
\begin{center}
\includegraphics[scale=0.6]{figure/chapter7/2.jpg}\\
\end{center}

\begin{framed}
THEOREM 3 \\
If $f$ is continuous on $[a,b]$, then there is some number $y$ in $[a,b]$ such that
\begin{equation}
f(y)>=f(x)
```

\end{equation}
for all x in $[a,b]$ (Figure 3).
\end{framed}
\begin{center}
\includegraphics[scale=0.8]{figure/chapter7/3.jpg}\\
\end{center}
\item These three theorems differ markedly from the theorems of Chapter 6.
\item The hypotheses of those theorems always involved continuity at a single point, while the hypotheses of the present theorems require continuity on a whole interval $[a,b]$

\item If continuity fails to hold at a single point, the conclusions may fail.
\item For example, let f be the function shown in Figure 4,

\begin{equation}
f(x) = \begin{cases}
-1 &0\leq x < \sqrt{2}\\
1 &\sqrt{2}\leq x\leq 2.\\
\end{cases}
\end{equation}
\begin{center}
\includegraphics[scale=0.8]{figure/chapter7/4.jpg}\\
\end{center}
\item Then f is continuous at every point of $[0,2]$ except $\sqrt{2}$, and
\begin{equation}
f(0)<0<f(2),
\end{equation}
but there is no point x in $[0,2]$ such that $f(x)=0$;
\bull{ the discontinuity at the single point $\sqrt{2}$ is sufficient to destroy the conclusion of Theorem 1.}

\item Similarly, suppose that f is the function shown in Figure 5,

\begin{equation}
f(x) = \begin{cases}
\frac{1}{x} &x\neq0\\
0 &x = 0,\\
\end{cases}
\end{equation}
\begin{center}
\includegraphics[scale=0.8]{figure/chapter7/5.jpg}\\
\end{center}

\item Then f is continuous at every point of $[0,1]$ except 0, but f is not bounded above on $[0,1]$.
\item In fact, for any number $N>0$ we have
\begin{equation}
f\bigg(\dfrac{1}{2N}\bigg)=2N>N.

\end{equation}

\item This example also shows that the closed interval $[a,b]$ in Theorem 2 cannot be replaced by the open interval (a,b), for the function f is continuous on $(0,1)$, but is not bounded there.
\item Finally, consider the function shown in Figure 6,
\begin{equation}
f(x) = \begin{cases}
x^2 &x<1\\
0 &x\geq 1.\\
\end{cases}
\end{equation}
\begin{center}
\includegraphics[scale=0.4]{figure/chapter7/6.jpg}\\
\end{center}

\begin{framed}
THEOREM 1 \\
If f is continuous on $[a,b]$ and
\begin{equation}
f(a)<0<f(b),
\end{equation}
then there is some x in $[a,b]$ such that \\
\begin{equation}
f(x)=0.
\end{equation}
\end{framed}

\begin{framed}
THEOREM 2 \\
If f is continuous on $[a,b]$, then f is bounded above on $[a,b]$, that is, there is some number N such that
\begin{equation}
f(x)<=N
\end{equation}
for all x in $[a,b]$.
\end{framed}

\begin{framed}
THEOREM 3 \\
If f is continuous on $[a,b]$, then there is some number y in $[a,b]$ such that
\begin{equation}
f(y)>=f(x)
\end{equation}
for all x in $[a,b]$ (Figure 3).
\end{framed}

\item On the interval $[0,1]$ the function f is bounded above, so f does satisfy the conclusion of Theorem 2
\item Even though f is not continuous on $[0,1]$.

\item But f does not satisfy the conclusion of Theorem 3
\begin{itemize}
\item ---there is no y in $[0,1]$ such that $f(y)>=f(x)$ for all x in $[0,1]$;
\item In fact, it is certainly not true that
\begin{equation}
f(1)>=f(x)
\end{equation}
for all x in $[0,1]$ so we cannot choose $y=1$, nor can we choose $0\leq
y<1$ because $f(y)<f(x)$ if x is any number with $y<x<1$.
\end{itemize}

\item This example shows that Theorem 3 is considerably stronger than Theorem 2.
\item Theorem 3 is often paraphrased by saying that a continuous function on a closed
interval ``takes on its maximum value" on that interval.
\item As a compensation for the stringency of the hypotheses of our three theorems,
the conclusions are of a totally different order than those of previous theorems.
\item They describe the behavior of a function, not just near a point, but on a whole
interval;
\bull{ such ``global" properties of a function are always significantly more difficult to
prove than ``local" properties, and are correspondingly of much greater power. }
\item To illustrate the usefulness of Theorems 1, 2, and 3, we will soon deduce some
important consequences, but it will help to first mention some simple generalizations
of these theorems.

\begin{framed}
THEOREM 4 \\
If f is continuous on $[a,b]$ and
\begin{equation}
f(a)<c<f(b),
\end{equation}
then there is some x in $[a,b]$ such that
\begin{equation}
f(x)=c.
\end{equation}

\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item Let $g=f-c$.
\item Then g is continuous, and g$(a)<0<g(b)$.
\item By Theorem 1, there is some x in $[a,b]$ such that $g(x)=0$. But this means
that $f(x)=c$.
\end{enumerate}
\end{proof}

\begin{framed}

THEOREM 5 \\
If f is continuous on $[a,b]$ and
\begin{equation}
f(a)>c>f(b),
\end{equation}
then there is some x in $[a,b]$ such that
\begin{equation}
f(x)=c.
\end{equation}
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item The function $-f$ is continuous on $[a,b]$ and $-f(a) < -c < -f(b)$.
\item By Theorem 4 there is some x in $[a,b]$ such that $-f(x) = -c$, which means that $f(x) = c$.
\end{enumerate}
\end{proof}

\item Theorems 4 and 5 together show that f takes on any value between $f(a)$ and $f(b)$. We can do even better than this: if c and d are in $[a,b]$, then f takes on any value between $f(c)$ and $f(d)$.
\item The proof is simple: if, for example, $c < d$, then just apply Theorems 4 and 5 to the interval $[c,d]$.
\item Summarizing, if a continuous function on an interval takes on two values, it takes on every value in between;
\bull{ this slight generalization of Theorem 1 is often called the Intermediate Value Theorem.}

\begin{framed}

THEOREM 6 \\
If f is continuous on $[a,b]$, then f is bounded below on $[a,b]$, that is, there is some number N such that $f(x)\leq N$ for all x in $[a,b]$.
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item The function $-f$ is continuous on $[a,b]$, so by Theorem 2 there is a number M such that $-f(x) \leq M$ for all x in $[a,b]$.
\item But this means that $f(x) \leq -M$ for all x in $[a,b]$, so we can let
\begin{equation}
N = -M.
\end{equation}
\end{enumerate}
\end{proof}

\begin{framed}
THEOREM 2 \\

If f is continuous on $[a,b]$, then f is bounded above on $[a,b]$, that is, there is some number N such that
\begin{equation}
f(x)<=N
\end{equation}
for all x in $[a,b]$.
\end{framed}
\item
\begin{itemize}
\item Theorems 2 and 6 together show that a continuous function f on $[a,b]$ is bounded on $[a,b]$, that is, there is a number N such that
\begin{equation}
|f(x)| \leq N
\end{equation}
for all x in $[a,b]$.
\item In fact, since Theorem 2 ensures the existence of a number N_1 such that
\begin{equation}
f(x) \leq N_1
\end{equation}
for all x in $[a,b]$, and Theorem 6 ensures the existence of a number N_2 such that
\begin{equation}
f(x) \leq N_2
\end{equation}
for all x in $[a,b]$, we can take
\end{itemize}
\begin{equation}
N = \max(|N_1|, |N_2|)
\end{equation}

\begin{framed}
THEOREM 7 \\
If f is continuous on $[a,b]$, then there is some y in $[a,b]$ such that $f(y) \leq f(x)$ for all x in $[a,b]$.
(A continuous function on a closed interval takes on its minimum value on that interval.)
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item The function $-f$ is continuous on $[a,b]$;
\bull{
by Theorem 3 there is some y in $[a,b]$ such that }
\begin{equation}
-f(y) \geq -f(x)
\end{equation}
for all x in $[a,b]$, which means that $f(y) \leq f(x)$ for all x in $[a,b]$.
\end{enumerate}
\end{proof}

\begin{framed}
THEOREM 8 \\
\begin{enumerate}[1.]
\item Every positive number has a square root.
\item In other words, if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.
\end{enumerate}
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item Consider the function
\begin{equation}
f(x) = x^2,
\end{equation}
which is certainly continuous.
\item Notice that the statement of the theorem can be expressed in terms of f:
\bull{ ``the number α has a square root'' means that f takes on the value α.}
\item The proof of this fact about f will be an easy consequence of Theorem 4.
\item There is obviously a number $b > 0$ such that $f(b) > \alpha$ (as illustrated in Figure 7);
\begin{center}
\includegraphics[scale=0.4]{figure/chapter7/7.jpg}\\
\end{center}
\item In fact, if $\alpha > 1$ we can take $b = \alpha$, while if $\alpha < 1$ we can take $b = 1$.
\item Since
\begin{equation}
f(0) < \alpha < f(b),
\end{equation}
Theorem 4 applied to $[0, b]$ implies that for some x (in $[0, b]$), we have
\begin{equation}
f(x) = \alpha,
\end{equation}
i.e.,
\begin{equation}
x^2 = \alpha.
\end{equation}
\end{enumerate}
\end{proof}

\item Precisely the same argument can be used to prove that a positive number has an nth root, for any natural number n.
\item If n happens to be odd, one can do better: every number has an nth root.
\item To prove this we just note that if the positive number alpha has the nth root x, i.e., if
\begin{equation}
x^n = \alpha,
\end{equation}

then
\begin{equation}
(-x)^n = - \alpha
\end{equation}
(since n is odd), so $- \alpha$ has the nth root $-x$. \item The assertion, that for odd n any number alpha has an nth root, is equivalent to the statement that the equation
\begin{equation}
x^n - \alpha = 0
\end{equation}
has a root if n is odd.
\item Expressed in this way the result is susceptible of great generalization.

\begin{framed}
THEOREM 9 \\
If n is odd, then any equation
\begin{equation}
x^n+a_{n-1}x^{n-1}+\dots+a_0 = 0
\end{equation}
has a root.
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item We obviously want to consider the function
\begin{equation}
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0;
\end{equation}
we would like to prove that f is sometimes positive and sometimes negative.
\item The intuitive idea is that for large $|x|$, the function is very much like
\begin{equation}
g(x) = x^n
\end{equation}
and, since n is odd, this function is positive for large positive x and negative for large negative x.
\item A little algebra is all we need to make this intuitive idea work.
\item The proper analysis of the function f depends on writing
\begin{equation}
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right).
\end{equation}
\item Note that

\begin{equation}
\left| \frac{a_{n-1}}{x}+\frac{a_{n-2}}{x^{2}}+\cdots +\frac{a_0}{x^n}\right| \leq \frac{a_{n-1}}{x}+\cdots+\frac{a_{0}}{x^{n}}
\end{equation}

```
\item Consequently, if we choose $x$ satisfying
\begin{equation}
(*) \quad |x|>1,2n|a_{n-1}|,\cdots,2n|a_0|,
\end{equation}
then
\begin{equation}
|x^k| > |x|
\end{equation}
and
\begin{equation}
\frac{a_{n-k}}{x^k} < \frac{a_{n-k}}{x} < \frac{a_{n-k}}{2n|a_{n-k}|} =
\frac{1}{2n},
\end{equation}
so
\begin{equation}
\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq
\frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.
\end{equation}
\item In other words,
\begin{equation}
- \frac{1}{2} \leq \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \leq \frac{1}{2},
\end{equation}
which implies that
\begin{equation}
\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}.
\end{equation}
\item Therefore, if we choose an $x_1>0$ which satisfies (*),then
\begin{equation}
\frac{(x_1)^n}{2} \leq (x_1)^n(1 + \frac{a_{n-1}}{x_1} + \cdots +
\frac{a_0}{(x_1)^n}) = f(x_1),
\end{equation}
so that $f(x_1)>0$.
\item On the other hand, if $x_2<0$ satisfies (*), then $(x_2)^n < 0$ and
\begin{equation}
\frac{(x_2)^n}{2} \leq (x_2)^n(1 + \frac{a_{n-1}}{x_2} + \cdots +
\frac{a_0}{(x_2)^n}) = f(x_2),
\end{equation}
so that $f(x_2) < 0$.
\item Now applying Theorem 1 to the interval $[x_2,x_1]$ we conclude that there is
an $x$ in $[x_2,x_1]$ such that $f(x) = 0$.
\end{enumerate}
\end{proof}

\begin{framed}
THEOREM 9 \\
If $n$ is odd, then any equation
\begin{equation}
x^n+a_{n-1}x^{n-1}+\dots+a_0 = 0
\end{equation}
has a root.
```

\end{framed}

\item Theorem 9 disposes of the problem of odd degree equations so happily that it would be frustrating to leave the problem of even degree equations completely undiscussed.
\item At first sight, however, the problem seems insuperable.
\item Some equations, like $x^2 - 1 = 0$, have a solution, and some, like $x^2 + 1 = 0$, do not---what more is there to say?
\item If we are willing to consider a more general question, however, something interesting can be said.
\item Instead of trying to solve the equation

\begin{equation}
x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,
\end{equation}
let us ask about the possibility of solving the equation
\begin{equation}
x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,
\end{equation}

\item for all possible numbers c.
\item This amount to allowing the constant term a_0 to vary.
\item The information which can be given concerning the solution of these equations depends on a fact which is illustrated in Figure 8.
\begin{center}
\includegraphics[scale=0.4]{figure/chapter7/8.jpg}\\
\end{center}
\item The graph of the function
\begin{equation}
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,
\end{equation}
with n even, contains, at least the way awe have drawn it, a lowest point.
\item In other words, there is a number y such that
\begin{equation}
f(y) \leq f(x)
\end{equation}
for all numbers x \bull{ --- the function f takes on a minimum value, not just on each closed interval, but on the whole line.}
\item (Notice that this is false if n is odd.)
\item The proof depends on Theorem 7, but a tricky application will be required.
\item
\begin{itemize}
\item We can apply Theorem 7 to any interval $[a,b]$, and obtain a point y_0 such that $f(y_0)$ is the minimum value of f on $[a,b]$;
\item but if $[a,b]$ happens to be the interval shown in Figure 8, for example, then the point y_0 will not be the place where f has its minimum value for the whole line.
\end{itemize}

\item In the next theorem the entire point of the proof is to choose an interval $[a,b]$ in such a way that this cannot happen.

```
\begin{framed}
THEOREM 10 \\
If $n$ is even and
\begin{equation}
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0
\end{equation}
then there is a number $y$ such that
\begin{equation}
f(y) \leq f(x)
\end{equation}
for all $x$.
\end{framed}


\begin{proof}
\begin{enumerate}[1.]\
\begin{framed}
THEOREM 9 \\
If $n$ is odd, then any equation
\begin{equation}
x^n+a_{n-1}x^{n-1}+\dots+a_0 = 0
\end{equation}
has a root.
\end{framed}

\item As in the proof of Theorem 9, if
\begin{equation}
M = \max(1 , 2n|a_{n-1}| , \cdots , 2n|a_0|),
\end{equation}
then for all $x$ with $|x| \geq M$, we have
\begin{equation}
\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}.
\end{equation}

\item Since $n$ is even, $x^n \geq 0$ for all $x$, so

\begin{equation}
\frac{x^n}{2} \leq x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)
= f(x),
\end{equation}
provided that $|x| > M$.
\item Now consider the number $f(0)$.
\item Let $b>0$ be a number such that
\begin{equation}
b^n > 2f(0) \text{ and also } b>M.
\end{equation}
\item Then, if $x \geq b$, we have (Figure 9)
\begin{center}
```

```
\includegraphics[scale=0.4]{figure/chapter7/9.jpg}\\
\end{center}
\begin{equation}
f(x) \geq \frac{x^n}{2} \geq \frac{b^n}{2} \geq f(0).
\end{equation}
\item Similarly, if $x \leq -b$, then
\begin{equation}
f(x) \geq \frac{x^n}{2} \geq \frac{(-b)^n}{2} = \frac{b^n}{2} \geq f(0).
\end{equation}
\item Summarizing: if
\begin{equation}
x \geq b \text{ or } x \leq -b
\end{equation}
then
\begin{equation}
f(x) \geq f(0).
\end{equation}
\item Now apply Theorem 7 to the function $f$ on the interval $[-b,b]$.
\item We conclude that there is a number $y$ such that (1) if
\begin{equation}
-b \leq x \leq b,
\end{equation}
then
\begin{equation}
f(x) \leq f(x).
\end{equation}
\item In particular,
\begin{equation}
f(y) \leq f(0).
\end{equation}
\item Thus (2) if
\begin{equation}
x \leq -b \text{ or } x \geq b
\end{equation}
then
\begin{equation}
f(x) \geq f(0) \geq f(y).
\end{equation}
\item Combining (1) and (2) we see that
\begin{equation}
f(y) \leq f(x)
\end{equation}
for all $x$.

\end{enumerate}
\end{proof}

\item Theorem 10 now allows us to prove thaw following result.

\begin{framed}
```

Theorem 11 \\
\begin{enumerate}[1.]
\item consider the equation
\begin{equation}
(*) \quad x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,
\end{equation}
and suppose n is even.

\item Then there is a number m such that (*) has a solution for $c \geq m$ and has no solution for $c<m$.
\end{enumerate}
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item Let
\begin{equation}
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ (Figure 10)}.
\end{equation}
\begin{center}
\includegraphics[scale=0.4]{figure/chapter7/10.jpg}\\
\end{center}
\item According to Theorem 10 there is a number y such that
\begin{equation}
f(y) \leq f(x)
\end{equation}
for all x.
\item Let $m = f(y)$.
\item If $c < m$, then the equation (*) obviously has no solution, since the left since the side always has a value $\geq m$.
\item If $c = m$, then (*) has y as a solution.
\item Finally, suppose $c>m$.
\item Let b be a number such that $b>y$ and $f(b)>c$.
\item Then
\begin{equation}
f(y) = m < c <f(b).
\end{equation}

\item Consequently, by Theorem 4, there is some number x in $[y,b]$ such that $f(x) = c$, so x is a solution of (*).
\end{enumerate}
\end{proof}

\item These consequences of Theorems 1, 2, and 3 are the only ones we will derive now \bull{(these theorems will play a fundamental role in everything we do later, however).}
\item Only one task remains--to prove Theorems 1, 2, and 3.
\item Unfortunately, we cannot hope to do this\bull{ --- on the basis of our present knowledge about the real numbers (namely, P1-P12) a proof is impossible.}
\begin{framed}

```
THEOREM 8 \\
\begin{enumerate}[1.]
\item Every positive number has a square root.
\item In other words, if $\alpha > 0$, then there is some number $x$ such that $x^2 =
\alpha$.
\end{enumerate}
\end{framed}
\item There are several ways of convincing ourselves that this gloomy conclusion is
actually the case.
\item For example, the proof of Theorem 8 relies only on the proof of Theorem 1;

\item If we could prove Theorem 1, then the proof of Theorem 8 would be complete,
and we would have a proof that every positive number has s square root.

\item As pointed out in Part I, it is impossible to prove this on the basis of P1-P12.
Again, suppose we consider the function

\begin{equation}
f(x) = \frac{1}{x^2-2}.
\end{equation}
\item If there were no number $x$ with
\begin{equation}
x^2 - 2,
\end{equation}
then $f$ would be continuous, since the denominator would never $= 0$. But $f$ is
not bounded on $[0,2]$.
\begin{framed}
THEOREM 1 \\
If $f$ is continuous on $[a,b]$ and
\begin{equation}
f(a)<0<f(b),
\end{equation}
then there is some $x$ in $[a,b]$ such that \\
\begin{equation}
f(x)=0.
\end{equation}
\end{framed}

\begin{framed}
THEOREM 2 \\
If $f$ is continuous on $[a,b]$, then $f$ is bounded above on $[a,b]$, that is, there is
some number $N$ such that
\begin{equation}
f(x)<=N
\end{equation}
for all $x$ in $[a,b]$.
\end{framed}

\begin{framed}
THEOREM 3 \\
```

If f is continuous on $[a,b]$, then there is some number y in $[a,b]$ such that
\begin{equation}
f(y)>=f(x)
\end{equation}
for all x in $[a,b]$ (Figure 3).
\end{framed}

\item So Theorem 2 depends essentially on the existence of numbers other than rational numbers, and therefore on some property of the real numbers other than P1-P12.

\item Despite our inability to prove Theorems 1, 2, and 3, they are certainly results which we want to be true,
\item If the pictures we have been drawing have any connection with the mathematics we are doing, if our notion of continuous function corresponds to any degree with our intuitive notion,
\item Theorems 1, 2, and 3 have got to be true.
\item Since a proof of any of these theorems must require some new property of $\mathbb{R}$ which has so far been overlooked, our present difficulties suggest a way to discover that property:\bull{ let us try to construct a proof of Theorem 1, for example, and see what goes wrong. }
\item One idea which seems promising is to locate the first point where $f(x) = 0$, that is, the smallest x in [a,b] such that $f(x) = 0$.
\item To find this point, first consider the set A which contains all numbers x in $[a,b]$ such that f is negative on $[a,x]$.
\item In Figure 11, x is such a point, while x' is not.
\item The set A itself is indicated by a heavy line.
\item Since f is negative at a, and positive at b, the set A contains some points greater than a, while all points sufficiently close to b are not in A.
\begin{center}
\includegraphics[scale=0.8]{figure/chapter7/11.jpg}\\
\end{center}
\item(We are here using the continuity of f on $[a,b]$, as well as Problem 6-15.)

\item Now suppose alpha is the smallest number which is greater than all members of A; clearly
\begin{equation}
a < \alpha < b.
\end{equation}

\item We claim that $f(\alpha) = 0$, and to prove this we only have to eliminate the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

\item Suppose first that $f(\alpha) < 0$.
\item Then, by Theorem 6-2, $f(x)$ would be less than 0 for all x in a small interval containing α,
\begin{itemize}
\item in particular for some numbers bigger than α (Figure 12); \item but this contradicts the fact that α is bigger than every member of A,
\item since the larger numbers would also be in A.

```
\end{itemize}
\item Consequently, $f(\alpha) < 0$ is false.
\begin{center}
\includegraphics[scale=0.5]{figure/chapter7/12.jpg}\\
\end{center}
\item On the other hand, suppose $f(\alpha) > 0$.
\item Again applying Theorem 6-2, we see that $f(x)$ would be positive for all $x$ in
a small interval containing $\alpha$,
\bull{ in particular for some numbers smaller than alpha (Figure 13).}
\begin{center}
\includegraphics[scale=0.5]{figure/chapter7/13.jpg}\\
\end{center}
\item This means that these smaller numbers are all not in $A$.
\item Consequently, one could have chosen an even smaller $\alpha$ which would be
greater than all members of $A$.
\item Once again we have a contradiction; $f(\alpha) > 0$ is also false.
\item Hence $f(\alpha) = 0$ and, we are tempted to say, Q.E.D.

\item We know, however, that something must be wrong, since no new properties of
$\mathbb{R}$ were ever used, and it does not require much scrutiny to find the
dubious point.
\item It is clear that we can choose a number alpha which is greater than all members
of $A$
\bull{ (for example, we can choose $\alpha = b$), but it is not so clear that we can
choose a smallest one.}
\item In fact, suppose $A$ consists of all numbers $x \geq 0$ such that $x^{2} < 2$.
\item If the number $\sqrt{2}$ did not exist, there would not be a least number greater
than all the members of $A$;\bull{ for any $y > \sqrt{2}$ we chose, we could always
choose a still smaller one.}
\end{enumerate}

\subsection{Problem}
\begin{enumerate}
\item
\begin{itemize}
\item For each of the following functions, decide which are bounded above or below
on the indicated interval, and which take on their maximum or minimum value.
\item (Notice that $f$ might have these properties even if $f$ is not continuous, and
even if the interval is not a closed interval.)
\end{itemize}
\begin{enumerate}
\item $f(x) = x^2$ on $(-1,1)$.
\item $f(x) = x^3$ on $(-1,1)$.
\item $f(x) = x^2$ on $\mathbb{R}$.
\item $f(x) = x^2$ on $[0,\infty)$.
\item $f(x) =
\begin{cases}
x^2 &x \leq a\\
a+2 &x >a,
\end{cases} $
```

on $(-a-1,a+1)$.\bull{(It will be necessary to consider several possibilities for a.)}
\item $f(x) =
\begin{cases}
x^2 &x \leq a\\
a+2 &x >a ,
\end{cases} $
on $[-a-1,a+1]$.
\item $f(x) =
\begin{cases}
0 &x \text{ irrational} \\
\dfrac{1}{q} &x = \frac{p}{q} \text{ in lowest terms}
\end{cases}$
on $[0,1].$
\item $f(x) =
\begin{cases}
1 &x \text{ irrational} \\
\dfrac{1}{q} &x = \dfrac{p}{q} \text{ in lowest terms}
\end{cases}$
on $[0,1]$.
\item $f(x) =
\begin{cases}
1 &x \text{ irrational} \\
\dfrac{1}{q} &x = \dfrac{p}{q} \text{ in lowest terms}
\end{cases}$
on $[0,1]$.
\item $f(x) =
\begin{cases}
1, &x \text{ irrational} \\
\dfrac{1}{q} &x = \dfrac{p}{q} \text{ in lowest terms}
\end{cases}$
on $[0,1]$.
\item $f(x) =
\begin{cases}
x &x \text{ rational} \\
0 &x \text{ irrational}
\end{cases}$
on $[0,a]$.
\item $f(x) = \sin^{2}\left(\cos x+\sqrt{a+a^{2}}\right)$ on $[0,a^3]$.
\item $f(x) = [x]$ on $[0,a]$.
\end{enumerate}

\item For each of the following polynomial functions f, find an integer n such that $f(x) = 0$ for some x between n and $n+1$.
\begin{enumerate}
\item $f(x) = x^{3}-x+3$.
\item $f(x) = x^{5}+5x^{4}+2x+1$.
\item $f(x) = x^{5}+x+1$.
\item $f(x) = 4x^{2}-4x+1$.
\end{enumerate}

\item Prove that there is some number x such that
\begin{enumerate}
\item $x^{179}+\frac{163}{1+x^{2}+\sin^{2}x} = 119$.
\item $\sin x = x-1$.
\end{enumerate}

\item This problem is a continuation of Problem 3-7.
\begin{enumerate}
\item If $n-k$ is even, and ≥ 0, find a polynomial function of degree n with exactly k roots.
\item
\begin{itemize}
\item A root a of the polynomial function f is said to have multiplicity m if
\begin{equation}
f(x) = (x-a)^{m}g(x),
\end{equation}
where g is a polynomial function that does not have a as a root.
\item Let f be a polynomial function of degree n.
\item Suppose that f has k roots, counting multiplicities, i.e., suppose that k is the sum of the multiplicities of all the roots.
\item Show that $n-k$ is even.
\end{itemize}
\end{enumerate}

\item
\begin{itemize}
\item Suppose that f is continuous on $[a,b]$ and that $f(x)$ is always rational.
\item What can be said about f ?
\end{itemize}

\item
\begin{itemize}
\item Suppose that f is a continuous function on $[-1,1]$ such that
\begin{equation}
x^{2}+(f(x))^{2} = 1
\end{equation}
for all x.
\item (This means that $(x,f(x))$ always lies on the unit circle.)
\item Show that either
\begin{equation}
f(x)=\sqrt{1-x^{2}}
\end{equation}
for all x, or else
\begin{equation}
f(x)=-\sqrt{1-x^{2}}
\end{equation}
for all x.

\end{itemize}

```
\item How many continuous function $f$ are there which satisfy
\begin{equation}
(f(x))^{2} = x^{2}
\end{equation}
for all $x$?
\end{enumerate}

%%%%%%%%
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%%%%%%%%
\newpage
\section{Least Upper Bound (p.131)}
\subsection{Context}
\begin{enumerate}
\item This chapter reveals the most important property of the real numbers.
\item Nevertheless, it is merely a sequel to Chapter 7;
\bull{ the path which must be followed has already been indicated, and further
discussion would be useless delay.}
\begin{framed}
\textbf{Definition}
\begin{itemize}
\item A set $A$ of real numbers is \textbf{bounded above} if there is a number
$x$ such that
\begin{equation}
x \geq a \text{ for every } a \text{ in } A.
\end{equation}

\item Such a number $x$ is called an \textbf{upper bound} for $A$.
\end{itemize}
\end{framed}
\item Obviously $A$ is bounded above if and only if
\begin{itemize}
\item there is a number $x$ which is an upper bound for $A$
\item(and in this case there will be lots of upper bounds for $A$);
\item we often say, as a concession to idiomatic English, that
\begin{center}
``$A$ has an upper bound"
\end{center}
\item when we mean that there is a number which is an upper bound for $A$.
\end{itemize}
\item Notice that the term ``bounded above" has now been used in two ways
\bull{
---first, in Chapter 7, in reference to functions, and now in reference to sets.}
\item This dual usage should cause no confusion, since it will always be clear whether
we are talking about a set of numbers or a function.
\item Moreover, the two definitions are closely connected:
\item if $A$ is the set
\begin{equation}
\{f(x): a < x < b\},
```

```
\end{equation}
then the function $f$ is bounded above on $[a, b]$ if and only if the set $A$ is
bounded above.

\item The entire collection $\mathbb{R}$ of real numbers, and the natural numbers
$\mathbb{N}$, are both examples of sets which are not bounded above.
\item An example of a set which is bounded above is
\begin{equation}
A = \{x: 0 < x < 1\}
\end{equation}
\item To show that $A$ is bounded above we need only name some upper bound for
$A$, which is easy enough;\bull{ for example, 138 is an upper bound for $A$, and so
are 2, 1$\dfrac{1}{2}$, 1$\dfrac{1}{4}$, and 1.}
\item Clearly, 1 is the least upper bound of $A$;
\begin{itemize}
\item although the phrase just introduced is self-explanatory,
\item in order to avoid any possible confusion
\item (in particular, to ensure that we all know what the superlative of ``less" means),
\item we define this explicitly.
\end{itemize}

\begin{framed}
A number $x$ is a least upper bound of $A$ if
\begin{enumerate}[1.]
\item $x$ is an upper bound of $A$
\item if $y$ is an upper bound of $A$, then $x \leq y$.
\end{enumerate}
\end{framed}
\item The use of the indefinite article ``a" in this definition was merely a concession
to temporary ignorance.
\item Now that we have made a precise definition, it is easily seen that if x and y are
both least upper bounds of $A$, then $x = y$.
\item Indeed, in this case
\begin{itemize}
\item $x \leq y$, since $y$ is an upper bound, and $x$ is a least upper bound.
\item $y \leq x$, since $x$ is am upper bound, and $y$ is a least upper bound.
\end{itemize}
\item It follows that $x=y$.
\item For this reason we speak of $\mathbf{the}$ the least upper bound of $A$.
\item The term $\mathbf{supermum}$ of $A$ is synonymous and has one advantage.
\item It abbreviates quite nicely to
\begin{center}
sup A ( pronounced ``soup A").
\end{center}
\item and saves us from the abbreviation
\begin{center}
lub A
\end{center}
(which is nevertheless used by some authors).
```

\item There is a series of important definitions, analogous to those just given, which can now be treated more briefly.
\item A set A of real numbers is $\mathbf{bounded below}$ if there is a number x such that
\begin{equation}
x \leq a \text{ for every a in A}.
\end{equation}

\item Such a number x is called a lower bound for A.
\item A number x is the greatest lower bound of A if
\begin{itemize}
\item x is a lower bound of A
\item if y is a lower bound of A, then $x \geq y$.
\end{itemize}

\item The greatest lower bound of A is also called the $\textbf{infimum}$ of A, abbreviated
\begin{center}
inf A;
\end{center}
some authors use the abbreviation
\begin{center}
glb A.
\end{center}
\item One detail has been omitted from our discussion so far
\bull{ --- the question of which sets have at least one, and hence exactly one, least upper bound or greatest lower bound.}
\item We will consider only least upper bounds, since the question for greatest lower bounds can then be answered easily (Problem 2).
\item If A is not bounded above, then A has no upper bound at all, so A certainly cannot be expected to have a least upper bound.
\item It is tempting to say that A does have a least upper bound if it has $\textit{some}$ upper bound, but, like the principle of mathematical induction, this assertion can fail to be true in a rather special way.
\item If $A = \emptyset$, then A is bounded above.
\item Indeed, any number x is an upper bound for $\emptyset$:

\begin{equation}
x \geq y \text{ for every } y \text{ in } \emptyset.
\end{equation}

\item simply because there is no y in $\emptyset$.
\item Since every number is an upper bound for $\emptyset$, there is surely no least upper bound for $\emptyset$.
\item With this trivial exception however, our assertion is true---and very important, definitely important enough to warrant consideration of details.
\item We are finally ready to state the last property of the real numbers which we need.
\begin{framed}
(Prop 13)\\

(The least upper bound property) \\
If A is a set of real numbers,
\begin{equation}
A \neq \emptyset,
\end{equation}
and A is bounded above, then A has a least upper bound.

\end{framed}
\item Property P13 may strike you as anticlimactic, but that is actually one of its virtues.
\item To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it.
\item Of course, the least upper bound property is not really so innocent as all that;
\bull{ after all, it does not hold for the rational numbers Q.}
\item For example, if A is the set of all rational numbers x satisfying
\begin{equation}
x^2 < 2,
\end{equation}

then there is no rational number y which is an upper bound for A and which is less than or equal to every other rational number which is an upper bond for A.
\item It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.
\begin{framed}

Theorem 7-1 (IVT)\\
If f is continuous on $[a, b]$ and
\begin{equation}
f(a) <0 < f(b),
\end{equation}
then there is some number x in $[a, b]$ such that
\begin{equation}
f(x) = 0.
\end{equation}

\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item Our proof is merely a rigorous version of the outline developed at the end of Chapter 7 \bull{ ---we will locate the smallest number x in $[a, b]$ with
\begin{equation}
f(x) = 0.
\end{equation}
}
\item Define the set A, shown in Figure 1, as follows:

```
\begin{equation}
A = \{x: a < x <b, \text{ and } f \text{ is negative on the interval }[a, x]\}
\end{equation}
\begin{center}
\includegraphics[scale=0.6]{figure/chapter8/1.jpg}\\
\end{center}
\item Clearly $A \neq \emptyset$, since $a$ is in $A$;
\begin{itemize}
\item in fact, there is some $\delta >0$ such that $A$ contains all points
$x$ satisfying
\begin{equation}
a < x < a+\delta;
\end{equation}
\item this follows from Problem 6-15, since $f$ is continuous on $[a, b]$ and $f(a)
<0$.
\end{itemize}

\item
\begin{itemize}
\item Similarly, $b$ is an upper bound for $A$ and,
\item in fact, there is a $\delta > 0$ such that all points $x$ satisfying
\begin{equation}
b-\delta < x < b
\end{equation}
are upper bounds for $A$;
\item this also follows from Problem 6-15, since $f(b) > 0$.
\end{itemize}

\item From these remarks it follows that $A$ has a least upper bound $\alpha$ and
that
\begin{equation}
a <\alpha <b.
\end{equation}
\item We now wish to show that $f(a) = 0$, by eliminating the possibilities
\begin{equation}
f(a) < 0\text{ and }f(a) > 0.
\end{equation}

\item Suppose first that $f(\alpha) < 0$.
\item By Theorem 6-3, there is a $ \delta > 0$ such that $f(x) < 0$ for
\begin{equation}
\alpha-\delta <x<\alpha +\delta
\end{equation}
(Figure 2).
\begin{center}
\includegraphics[scale=0.8]{figure/chapter8/2.jpg}\\
\end{center}
\item Now there is some number $x_0$ in $A$ which satisfies
\begin{equation}
\alpha -\delta < x_0 <\alpha
```

\end{equation}
\bull{ (because otherwise α would not be the least upper bound of A).}
\item This means that f is negative on the whole interval $[a, x_0]$.
\item But if x_1 is a number between α and $\alpha +\delta$, then f is also negative on the whole interval $[x_0, x_1]$.
\item Therefore f is negative on the interval $[a, x_1]$, so x is in A.
\item But this contradicts the fact that α is an upper bound for A;
\bull{ our original assumption that $f(\alpha) <0$ must be false.}
\item Suppose, on the other hand, that $f(\alpha) > 0$.

\item Then there is a number $\delta >0$ such that $f(x) > 0$ for $\alpha -\delta <x<\alpha +\delta$ (Figure 3).
\begin{center}
\includegraphics[scale=0.7]{figure/chapter8/3.jpg}\\
\end{center}
\item Once again we know that there is an x_0 in A satisfying
\begin{equation}
\alpha - \delta < x_0 <\alpha;
\end{equation}
but this means that f is negative on $[a, x_0]$, which is impossible, since $f(x_0)>0$
\item Thus the assumption $f(\alpha) > 0$ also leads to a contradiction, leaving $f(\alpha) = 0$ as the only possible alternative.
\end{enumerate}
\end{proof}
\item The proofs of Theorems 2 and 3 of Chapter 7 require a simple preliminary result, which will play much the same role as Theorem 6-3 played in the previous proof.

\begin{framed}
Theorem 1 \\
If f is continuous at a, then there is a number $\delta >0$ such that f is bounded above on the interval
\begin{equation}
(a -\delta, a +\delta)
\end{equation}
(see Figure 4).
\end{framed}
\begin{center}
\includegraphics[scale=0.5]{figure/chapter8/4.jpg}\\
\end{center}
\begin{proof}\
\begin{enumerate}[1.]
\item Since $\dislim_{x \to a}{f(x)}=f(a)$, there is, for every $\varepsilon>0$, a $\delta>0$ such that, for all x, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then

\begin{equation}
|f(x)-f(a)|<\varepsilon.
\end{equation}
\item It is only necessary to apply this statement to some particular ε (any one will do), for example,
\begin{equation}
\varepsilon = 1 .
\end{equation}
\item We conclude that there is a $\delta>0$ such that, for all x, if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-f(a)|<1
\end{equation}
\item It follows, in particular, that if
\begin{equation}
|x-a|<\delta,
\end{equation}
then
\begin{equation}
|f(x)-f(a)|<\varepsilon.
\end{equation}
\item This completes the proof:
\bull{ on the interval
\begin{equation}
(a-\delta,a+\delta)
\end{equation}
the function f is bounded above by $f(a) +1$.}
\end{enumerate}
\end{proof}
\item It should hardly be necessary to add that we can now also prove that f is bounded below on some interval
\begin{equation}
(a-\delta,a+\delta),
\end{equation}
and, finally, that f is bounded on some open interval containing a.
\item A more significant point is the observation that if
\begin{equation}
\dislim_{x \to a^+}{f(x)}=f(a),
\end{equation}
then there is a $\delta > 0$ such that f is bounded on the set
\begin{equation}
\{x: a \leq x < a+\delta\},
\end{equation}
and a similar observation holds if
\begin{equation}
\dislim_{x \to b^-}{f(x)}=f(b).
\end{equation}

\item Having made these observations (and assuming that you will supply the proofs), we tackle our second major theorem.

\begin{framed}

Theorem 7-2 \\
If f is continuous on $[a,b]$, then f is bounded above on $[a,b]$.
\end{framed}
\begin{proof}\
\begin{enumerate}[1.]
\item Let
\begin{equation}
A = \{x:a \leq x \leq b \text{ and } f \text{ is bounded above on } [a,x]\}
\end{equation}
\item Clearly $A \neq \emptyset$ (since a is in A), and A is bounded above (by b), so A has a least upper bound α.
\item Notice that we are here applying the term ``bounded above" both to the set A,
\begin{itemize}
\item which can be visualized as lying on the horizontal axis, and to f,
\item i.e., to the sets
\begin{equation}
\{f(y):a\leq y \leq x \},
\end{equation}
which can be visualized as lying on the vertical axis (Figure 5).
\end{itemize}

\begin{center}
\includegraphics[scale=0.8]{figure/chapter8/5.jpg}\\
\end{center}
\item Our first step is to prove that we actually have
\begin{equation}
a = b.
\end{equation}
\item Suppose, instead, that
\begin{equation}
\alpha<b.
\end{equation}
\item By Theorem 1 there is $\delta > 0$ such that f is bounded on
\begin{equation}
(\alpha-\delta,\alpha+\delta).
\end{equation}
\item Since α is the least upper bound of A there is some x_0 in A satisfying
\begin{equation}
\alpha-\delta < x_0 <\alpha.
\end{equation}
\item This means that f is bounded on $[a, x_0]$.
\item But if x_1 is any number with
\begin{equation}

```
\alpha-\delta < x_1 <\alpha,
\end{equation}
then $f$ is also bounded on $[x_0, x_1]$.
\item Therefore $f$ is bounded on $[a, x_1]$, so $x_1$ is in $A$, contradicting the
fact that $\alpha$ is an upper bound for $A$.
\item This contradiction shows that
\begin{equation}
a = b.
\end{equation}
\item One detail should be mentioned:
\begin{itemize}
\item this demonstration implicitly assumed that $a <\alpha$ [ so that $f$ would be
defined on some interval
\begin{equation}
(\alpha-\delta,\alpha+\delta);
\end{equation}
the possibility a = $\alpha$ can be ruled out similarly, using the existence of a
$\delta > 0$ such that f is bounded on
\begin{equation}
\{x:a\leq x < a+\delta \}.
\end{equation}
\end{itemize}
\item The proof is not quite complete\bull{---we only know that $f$ is bounded on
$[a, x]$ for every $x <b$, not necessarily that $f$ is bounded on $[a, b]$.}
\item However, only one small argument needs to be added.
\item There is a $\delta > 0$ such that $f$ is bounded on
\begin{equation}
\{x: b-\delta < x \leq b\}.
\end{equation}
\item There is $x_0$ in $A$ such that
\begin{equation}
b-\delta < x_0 <b.
\end{equation}
\item Thus $f$ is bounded on $[a, x_0]$ and also on $[x_0, b]$, so $f$ is bounded on
$[a, b]$.
\end{enumerate}
\end{proof}

\item To prove that third important theorem we resort a trick.
\begin{framed}
Theorem 7-3 \\
If $f$ is continuity on $[a,b]$, then there is a number y in $[a,b]$ such that $f(y)\geq
f(x)$ for all $x$ in $[a,b]$.
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item We already know that $f$ is bounded on $[a, b]$, which means that the set
\begin{equation}
\{f(x): x \text{ in } [a, b]\}
```

```
\end{equation}
is bounded.
\item This set is obviously not $\emptyset$, so it has a least upper bound $\alpha$.
\item Since
\begin{equation}
\alpha \geq f(x)
\end{equation}
for $x$ in $[a, b]$ it suffices to show that
\begin{equation}
a = f(y)
\end{equation}
for some $y$ in $[a, b]$.
\item Suppose instead that
\begin{equation}
\alpha \neq f(y)
\end{equation}
for all $y$ in $[a, b]$.
\item Then the function g defined by
\begin{equation}
g(x) = \dfrac{1}{\alpha-f(x)}\text{, } x \text{ in } [a,b]
\end{equation}
is continuous on $[a,b]$, since the denominator of the right side is never 0.
\item On the other hand, $\alpha$ is the least upper bound of
\begin{equation}
\{f(x):x \text{ in }[a,b]\};
\end{equation}
this means that for every $\varepsilon>0$ there is $x$ in $[a,b]$ with
\begin{equation}
\alpha-f(x)<\varepsilon
\end{equation}
\item This, in turn, means that for every $\varepsilon>0$ there is $x$ in $[a,b]$ with
\begin{equation}
g(x)>\dfrac{1}{\varepsilon}.
\end{equation}
\item But \textit{this} means that $g$ is not bounded on $[a,b]$, contradiction the
previous theorem.
\end{enumerate}
\end{proof}

\item At the beginning of this chapter the set of natural numbers $\mathbb{N}$ was
given as an example of an unbounded set.
\item We are now going to prove that $\mathbb{N}$ is unbounded.
\item After the difficult theorems proved in this chapter you may be startled to find
such an ``obvious" theorem winding up our proceedings.
\item If so, you are, perhaps, allowing the geometrical picture of $\mathbb{R}$ to
influence you too strongly.

\item ``Look, "you may say,`` the real numbers
\begin{center}
\includegraphics[scale=0.7]{figure/chapter8/untitle1.jpg}\\
```

```
\end{center}
\begin{center}
Figure
\end{center}
so every number $x$ is between two integers $n, n+1$ (unless $x$ is itself an
integer).
\item Basing the argument on a geometric picture is not a proof, however, and even
the geometric picture contains an assumption:
\bull{ that if you place unit segments end-to-end you will eventually get a segment
larger than any given segment.}
\item This axiom, often omitted from a first introduction to geometry, is usually
attributed (not quite justly) to Archimedes, and the corresponding property for
numbers, that $\mathbb{N}$ is not bounded, is called the \textit{Archimedian
property} of the real numbers.
\item This property is not a consequence of P1-P12 (see reference [17] of the
Suggested Reading), although it does hold for $\mathbb{Q}$, of course.
\item Once we have P13 however, there are no longer any problems.


\begin{framed}
Theorem 2 \\
$\mathbb{N}$ is not bounded above.
\end{framed}

\begin{proof}\
\begin{enumerate}[1.]
\item Suppose $\mathbb{N}$ were bounded above.
\item Since
\begin{equation}
\mathbb{N} \neq \emptyset,
\end{equation}
there would be a least upper bound $\alpha$ for $\mathbb{N}$.
\item Then
\begin{equation}
\alpha \geq n \text{ for all } n \text{ in } \mathbb{N}
\end{equation}
\item Consequently,
\begin{equation}
\alpha \geq n \text{ for all } n \text{ in } \mathbb{N}
\end{equation}
since $n + 1$ is in $\mathbb{N}$ if $n$ is in $\mathbb{N}$.
\item But this means that
\begin{equation}
\alpha -1 \geq n \text{ for all } n \text{ in } \mathbb{N}
\end{equation}
and this means that $\alpha-1$ is also an upper bound for $\mathbb{N}$,
contradicting the fact that $\alpha$ is the least upper bound.

\end{enumerate}
\end{proof}
```

\item There is a consequence of Theorem 2 (actually an equivalent formulation)
which we have very often assumed implicitly.
\begin{framed}
For every $\varepsilon>0$ there is a natural number n with
$\dfrac{1}{n}<\varepsilon$.
\end{framed}
\begin{proof}\
\begin{enumerate}[1.]
\item Suppose not; then
\begin{equation}
\dfrac{1}{n} \geq \varepsilon
\end{equation}
for all n in $\mathbb{N}$.
\item Thus
\begin{equation}
n \leq \dfrac{1}{\varepsilon}
\end{equation}
for all n in $\mathbb{N}$.
\item But this means that $\dfrac{1}{\varepsilon}$ is an upper bound for
$\mathbb{N}$, contradicting Theorem 2.
\end{enumerate}
\end{proof}
\item A brief glance through Chapter 6 will show you that the result of Theorem 3
was used in the discussion of many examples.
\item Of course, Theorem 3 was not available at the time, but the examples were so
important that in order to give them some cheating was tolerated.
\item As partial justification for this dishonesty we can claim that this result was never
used in the proof of a theorem, but if your faith has been shaken, a review of all the
proofs given so far is in order.
\item Fortunately, such deception will not be necessary again.
\item We have now stated every property of the real numbers that we will ever need.
\item Henceforth, no more lies.
\end{enumerate}

\subsection{Problem}
\begin{enumerate}
\item
\begin{itemize}
\item Find the least upper bound and the greatest lower bound (if they exist) of the
following sets.
\item Also decide which sets have greatest and least elements (i.e., decide when the
least upper bound and greatest lower bound and greatest lower bound happens to
belong to the set)
\end{itemize}

\begin{enumerate}
\item $\bigg\{\dfrac{1}{n}:n \text{ in }\mathbb{N}\bigg\}$.
\item $\bigg\{\dfrac{1}{n}:n \text{ in }\mathbb{Z} \text{ and } n \neq 0. \bigg\}$.

```
\item $\{x:x=0 \text{ or } x = \dfrac{1}{n} \text{ for some } n
\text{ in }\mathbb{N}\}$.
\item $\{x:0\leq x \leq \sqrt{2} \text{ and } x \text{ is rational}\}$.
\item $\{x:x^2+x+1 \geq 0\}$.
\item $\{x:x^2+x-1 < 0\}$.
\item $\{x:x<0 \text{ and } x^2+x-1 < 0 \}$.
\item $\bigg\{\dfrac{1}{n}+(-1)^n:n \text{ in }\mathbb{N}\bigg\}$.
\end{enumerate}
\item
\begin{enumerate}
\item
\begin{itemize}
\item Suppose $A\neq\emptyset$ is bounded below.
\item Let $-A$ denote the set of all $-x$ for $x$ in $A$.
\item Prove that $-A \neq \emptyset$ is bounded above, and that $-\sup(-A)$ is the
greatest lower bound of $A$.
\end{itemize}
\item
\begin{itemize}
\item If $A \neq \emptyset$ is bounded below, let $B$ be the set of all lower bounds
of $A$.
\item Show that $B\neq\emptyset$, that $B$ is bounded above, and that $\sup B$ is
the greatest lower bound of $A$.

\end{itemize}
\end{enumerate}

\item Let $f$ be a continuous function on [a,b] with $f(a)<0<f(b)$.
\begin{enumerate}
\item
\begin{itemize}
\item The proof of the Theorem 1 showed that there is a smallest $x$ in $[a,b]$ with
$f(x)= 0$.
\item Is there necessarily a second smallest $x$ in $[a,b]$ with$f(x)=0$?
\item Shoe that there is a largest $x$ in $[a,b]$ with $f(x)=0$.
\item (Try to give an easy proof by considering a new function closely related to $f$.)
\end{itemize}
\item
\begin{itemize}
\item The proof of Theorem 1 depend upon consideration of
\begin{equation}
A=\{x:a\leq x \leq b \text{ and } f \text{ is negative on }[a,b] \}.
\end{equation}
\item Give another proof of Theorem 1, which depends on consideration of
\begin{equation}
A=\{x:a\leq x \leq b \text{ and } f(x) < 0 \}.
\end{equation}
\item Which point $x$ in $[a,b]$ with $f(x)=0$ will this proof locate? \item Give an
example where the sets $A$ and $B$ are not the same.
\end{itemize}
```

```
\end{enumerate}
\item Suppose that $f$ is continuous on $[a, b]$ and that $f(a) = f(b) = 0$.
\begin{enumerate}
\item
\begin{itemize}
\item Suppose also that $f(x_0) > 0$ for some $x_0$ in $[a, b]$.
\item Prove that there are numbers $c$ and $d$ with a
\begin{equation}
a \leq c < x_0 < d \leq b
\end{equation}
such that
\begin{equation}
f(0) = f(d) = 0,
\end{equation}
but $f(x) > 0$ for all $x$ in $(c,d)$.
\item Hint: The previous problem can be used to good advantage.
\end{itemize}

\item
\begin{itemize}
\item Suppose that $f$ is continuous on $[a, b]$ and that
\begin{equation}
f(a) < f(b).
\end{equation}
\item Prove that
there are numbers $c$ and $d$ with
\begin{equation}
a \leq c < d \leq b
\end{equation}
such that $f(c) = f(a)$ and $f(d) = f(b)$ and
\begin{equation}
f(a) < f(x) < f(d)
\end{equation}
for all $x$ in $(c,d)$.
\end{itemize}

\end{enumerate}

\item \begin{enumerate}
\item
\begin{itemize}
\item Suppose that $y - x > 1$.
\item Prove that there is an integer $k$ such that $x < k < y$.
\item Hint: Let $l$ by the largest integer satisfying $l < x$, and consider $l + 1 $.
\end{itemize}

\item
\begin{itemize}
\item Suppose $x < y$.
```

\item Prove that there is a rational number r such that $x < r<y$. \item Hint: If
\begin{equation}
\dfrac{1}{n} < y - x,
\end{equation}
then
\begin{equation}
ny-nx > 1.
\end{equation}
\item (Query: Why have parts (a) and (b) been postponed until this problem set?)
\end{itemize}
\item
\begin{itemize}
\item Suppose that $r < s$ are rational numbers.
\item Prove that there is an irrational number between r and s. \item Hint: As a start, you know that there is an irrational number between 0 and 1.

\end{itemize}
\item
\begin{itemize}
\item Suppose that $x < y$.
\item Prove that there is an irrational number between x and y. \item Hint: It is unnecessary to do any more work; this follows from (b) and (c).
\end{itemize}

\end{enumerate}

\end{enumerate}

\newpage
\begin{appendices}
\section{Uniform Continuity}
\begin{enumerate}
\item Now that we've come to the end of the ``foundations," it might be appropriate to slip in one further fundamental concept.
\item This notion is not used crucially in the rest of the book, but it can help clarify many points later on.
\item We know that the function
\begin{equation}
f(x) = x^2
\end{equation}
is continuous at a for all a.
\item In other words, if a is any number, then for every $\varepsilon > 0$ there is some
$\delta > 0$ such that, for all x, if
\begin{equation}
|x - a| <\delta,
\end{equation}
then
\begin{equation}
|x^2 -a^2| <\varepsilon.

\end{equation}
\item Of course, δ depends on ε.
\item But δ also depends on a\bull{--- the δ that works at a might not work at b (Figure 1).}
\item Indeed, it's clear that given ε there is no one $\delta>0$ that works for all a, or even for all positive a.
\item In fact, the number $a + \dfrac{\delta}{2}$ will certainly satisfy
\begin{equation}
|x-a| <\delta,
\end{equation}
but if $a > 0$, then
\begin{equation}
\bigg|(a+\dfrac{\delta}{2})^2 - a^2 \bigg| = \bigg|a\delta+\dfrac{\delta^2}{4}\bigg|
\geq a \delta
\end{equation}
and this won't be $<\varepsilon$ once $a > \dfrac{\varepsilon}{\delta}$.
\item (This is just an admittedly confusing computational way of saying that f is growing faster and faster!
\item On the other hand, for any $\varepsilon>0$ there will be one $\delta > 0$ that works for all a in any interval $[-N, N)$.
\item In fact, the δ which works at N or $-N$ will also work everywhere else in the interval.
\item As a final example, consider the function
\begin{equation}
f(x) = \sin\bigg(\dfrac{1}{x}\bigg),
\end{equation}
or the function whose graph appears in Figure 18 on page 62.
\item It is easy to see that, so long as $\varepsilon < 1$, there will not be one $\delta >0$ that works for these functions at all points a in the open interval $(0, 1)$.
\item These examples illustrate important distinctions between the behavior of various continuous functions on certain intervals, and there is a special term to signal this distinction.

\begin{framed}
\textbf{DEFINITION}\\
The function f is uniformly continuous on an interval A if for every $\varepsilon > 0$ there is some $\delta >0$ such that, for all x and y in A, if
\begin{equation}
|x - y| < \delta,
\end{equation}
then
\begin{equation}
|f(x) - f(y)|< \varepsilon.
\end{equation}
\end{framed}

\item We've seen that a function can be continuous on the whole line, or on an open interval, without being uniformly continuous there.
\item On the other hand, the function
\begin{equation}

```
f(x) = x^2
\end{equation}
did turn out to be uniformly continuous on any closed interval.
\item This shouldn't be too surprising it's the same sort of thing that occurs when we
ask whether a function is bounded on an interval\bull{--- and we would be led to
suspect that any continuous function on a closed interval is also uniformly continuous
on that interval.}
\item In order to prove this, we'll need to deal first with one subtle point.
\item Suppose that we have two intervals $[a, b]$ and $[b, c)$ with the common
endpoint $b$, and a function $f$ that is continuous on $(a,c)$. \item Let
$\varepsilon > 0$ and suppose that the following two statements hold:
\begin{enumerate}[I.]
\item if $x$ and $y$ are in $[a, b]$ and $|x-y| < \delta_1$, then $|f(x) - f(y)| <
\varepsilon$,
\item if $x$ and $y$ are in $[b, c]$ and $|x – y| < \delta_2$, then $|f(x) - f(y)|<
\varepsilon$.
\end{enumerate}
\item We'd like to know if there is some $\delta >0$ such that
\begin{equation}
|f(x) - f(y)|< \varepsilon
\end{equation}
whenever $x$ and $y$ are points in $[a, c]$ with $|x - y| < \delta$.
\item Our first inclination might be to choose $\delta$ as the minimum of
$\delta_1$ and $\delta_2$.
\item But it is easy to see what goes wrong (Figure 2):\bull{ we might have $x$ in
$[a, b]$ and $y$ in $[b, c)$, and then neither (I) nor (II) tells us anything about
\begin{equation}
|f(x) - f(y)|.
\end{equation}
}
\item So we have to be a little more cagey, and also use continuity of $f$ at $b$.

\begin{framed}
LEMMA \\ \
\begin{enumerate}[1.]
\item Let $a<b<c$ and let $f$ be continuous on the interval $[a,c]$.
\item Let $\varepsilon > 0$, and suppose that statement (I) and (II) hold.
\item Then there is a $\delta > 0$ such that, if
\begin{equation}
|x-b| < \delta_{3}
\end{equation}
then
\begin{equation}
|f(x) - f(y)| < \varepsilon.
\end{equation}
\end{enumerate}
\end{framed}

\begin{proof}\
```

```
\begin{enumerate}[1.]
\item Since $f$ is continuous at $b$, there is a $\delta_{3} > 0$ such that, if
\begin{equation}
|x-b|<\delta_{3},
\end{equation}
then
\begin{equation}
|f(x)-f(y)|< \dfrac{\varepsilon}{2}.
\end{equation}

\item It follows that (III) if
\begin{equation}
|x-b|<\delta_{3} \text{ and } |y-b|<\delta_{3}
\end{equation}
then
\begin{equation}
|f(x)-f(y)|<\varepsilon .
\end{equation}

\item Choose $\delta$ to be the minimum of $\delta_{1}$ , $\delta_{2}$ , and
$\delta_{3}$.
\item We claim that this $\delta$ works.
\item In fact, suppose that $x$ and $y$ are both in $[b,c]$, then
\begin{equation}
|f(x)-f(y)<\varepsilon|
\end{equation}
by (II).
\item The only other possibility is that
\begin{equation}
x<b<y \quad y<b<x.
\end{equation}

\item In either case, since
\begin{equation}
|x-y|<\delta,
\end{equation}
we also have
\begin{equation}
|x-b|<\delta \text{ and } |y-b|<\delta.
\end{equation}

\item So
\begin{equation}
|f(x)-f(y)|<\varepsilon
\end{equation}
by(III)
\end{enumerate}
\end{proof}
```

```latex
\begin{framed}
THEOREM 1 \\
If $f$ is continuous on $[a,b]$, then $f$ is uniformly continuous on $[a,b]$.
\end{framed}
\begin{proof}\
\begin{enumerate}[1.]
\item It's the usual trick, but we've got to be a little bit careful about the mechanism of
the proof.
\item For $\varepsilon > 0$ let's say that $f$ is $\varepsilon-good$ on $[a,b]$ if there
is some $\delta>0$ such that, for all $y$ and $z$ in $[a,b]$, if

\begin{equation}
|y-z|<\delta,
\end{equation}
then
\begin{equation}
|f(y)-f(z)|<\varepsilon.
\end{equation}

\item Then we're trying to prove that $f$ is $\varepsilon-good$ on $[a,b]$ for all
$\varepsilon>0$.
\item Consider any particular $\varepsilon > 0$.
\item Let
\begin{equation}
A = {x:a \leq x \leq b \text{ and } f \text{ is }\varepsilon-good \text{ on } [a,x]}.
\end{equation}


\item Then $A\neq \emptyset$ (since $a$ is in $A$), and $A$ is bounded above (by
$b$), so $A$ has a least upper bound $\alpha$.
\item We really should write $\alpha_{\varepsilon}$, since $A$ and $\alpha$ might
depend on $\varepsilon$.
\item But we won't since we intend to prove that $\alpha = b$, no matte what
$\varepsilon$ is.
\item Suppose that we had $\alpha < b$.
\item Since $f$ is continuous at $\alpha$, there is some $\delta_{0} > 0$ such that, if
\begin{equation}
|y-\alpha|<\delta_{0},
\end{equation}
then
\begin{equation}
|f(y)-f(\alpha)| < \dfrac{\varepsilon}{2}.
\end{equation}
\item Consequently, if
\begin{equation}
|y-\alpha| < \delta_{0} \text{ and }|z-\alpha| < \delta_{0},
\end{equation}
then
\begin{equation}
|f(y)-f(z)|< \varepsilon.
```

```
\end{equation}

\item So $f$ is surely $\varepsilon - good$ on the interval
\begin{equation}
[\alpha - \delta_{0},\alpha + \delta_{0}].
\end{equation}

\item On the other hand, since $\alpha$ is the least upper bound of $A$, it
$\varepsilon - good$ on
\begin{equation}
[a,a+\delta_{0}],
\end{equation}
so $a + \delta_{0}$ is in $A$, contradiction the fact that $\alpha$ is an upper bound.

\item To complete the proof we just have to show that $\alpha = b$ is actually in $A$.
\item The argument for this is practically the same:
\item Since $f$ is continuous at $b$, there is some $\delta_{0} > 0$ such that, if
\begin{equation}
|b-y|<\delta_{0},
\end{equation}
then
\begin{equation}
|f(y)-f(b)| < \dfrac{\varepsilon}{2}.
\end{equation}
\item So $f$ is $\varepsilon - good$ on
\begin{equation}
[b-\delta_{0},b].
\end{equation}
\item But $f$ is also $\varepsilon - good$ on
\begin{equation}
[a,b- \delta_{0}],
\end{equation}
so the Lemma implies that $f$ is $\varepsilon - good$ on $[a,b]$.
\end{enumerate}
\end{proof}

\end{enumerate}
\end{appendices}

\end{document}
```

<
o
(i)
(ii)
a
X
b
c

FIGURE 2
LEMMA
Let a < b < c and let f be continuous on the interval (a, c). Let € > 0, and suppose that statements (i) and (ii) hold. Then there is a d > 0 such that,
if x and y are in [a, c) and (x - y < 8, then if(x) - f(y)< €.

%%%%%%%%%%%%%%%%%%%%%%%%%
(d) Suppose that x < y. Prove that there is an irrational number between x and y. Hint: It is unnecessary to do any more work; this follows from
(b) and (c). *6. A set A of real numbers is said to be dense if every open interval contains a
point of A. For example, Problem 5 shows that the set of rational numbers and the set of irrational numbers are each dense. (a) Prove that if f is continuous and f(x) = 0 for all numbers x in a dense
set A, then f(x) = 0 for all x. (b) Prove that if f and g are continuous and f(x) = g(x) for all x in a dense
set A, then f(x) = g(x) for all x. (c) If we assume instead that f(x) > 8(x) for all x in A, show that f(x) >
8(x) for all x. Can > be replaced by > throughout?
Prove that if f is continuous and f(x + y) = f(x) + f(y) for all x and y, then there is a number c such that f(x) = cx for all x. (This conclusion can be demonstrated simply by combining the results of two previous problems.) Point of information: There do exist noncontinuous functions f satisfying f(x + y) = f(x) + f(y) for all x and y, but we cannot prove this now; in fact, this simple question involves ideas that are usually never mentioned in any undergraduate course. The Suggested Reading contains references.
7.

8. Least Upper Bounds 139
*8.
Suppose that f is a function such that f(a) = f(b) whenever a < b (Figure 6).
ra"
X-at
(a) Prove that lim f(x) and lim f(x) both exist. Hint: Why is this prob
lem in this chapter? (b) Prove that f never has a removable discontinuity (this terminology comes
from Problem 6-16). (c) Prove that if f satisfies the conclusions of the Intermediate Value The
orem, then f is continuous.
*9.
If f is a bounded function on [0, 1], let III f ||| = sup{if(x) : x in [0, Prove analogues of the properties of || || in Problem 7-14.
10. Suppose a > 0. Prove that every number x can be written uniquely in the
form x = ka + x', where k is an integer, and 0 < x <a.
FIGURE 6
one side of PLAN
sides of P
T
11. (a) Suppose that aj, aj, a3, ... is a sequence of positive numbers with
Ant1 < an/2. Prove that for any € > Othere is some n with an <€. (b) Suppose P is a regular polygon inscribed inside a circle. If p' is the
inscribed regular polygon with twice as many sides, show that the difference between the area of the circle and the area of p' is less than half the
difference between the area of the circle and the area of P (use Figure 7). (c) Prove that there is a regular polygon P inscribed in a circle with area
as close as desired to the area of the circle. In order to do part (C) you will need part (a). This was clear to the Greeks, who used part (a) as the basis for their entire treatment of proportion and area. By calculating the areas of polygons, this method ("the method of exhaustion") allows computations of a to any desired accuracy, Archimedes used it to show
that 273 < i < 27. But it has far greater theoretical importance: *d) Using the fact that the areas of two regular polygons with the same num
ber of sides have the same ratio as the square of their sides, prove that the areas of two circles have the same ratios as the square of their radii. Hint: Deduce a contradiction from the assumption that the ratio of the areas is greater, or less, than the ratio of the square of the radii by inscribing appropriate polygons.
FIGURE 7
12. Suppose that A and B are two nonempty sets of numbers such that x = y
for all x in A and all y in B.
(a) Prove that sup A Sy for all y in B. (b) Prove that sup A sinf B.
13. Let A and B be two nonempty sets of numbers which are bounded above, and
let A+B denote the set of all numbers x+y with x in A and y in B. Prove that
sup(A+B) = sup A+sup B. Hint: The inequality sup(A+B) < sup Atsup B is easy. Why? To prove that sup A +sup B 5 sup(A + B) it suffices to prove that sup A + sup B < sup(A+B) + ε for all € > 0; begin by choosing x in A

注： 本附件內容為每名學生可各自自由調整，通過前面的 latex。只是這裡轉化為 pdf檔。

Spivak

January 15, 2019

Contents

5 Limit (p.91)

5.1 Context

1. The concept of a limit is surely the most important, and probably the most difficult one in all calculus.

2. The goal of this chapter is

   "the definition of limits,"

   but we are, once more, going to begin with a provisional definition;

   - what we shall define is not the word "limit" but the notion of a function approaching a limit.

PROVISIONAL DEFINITION

The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be

- sufficiently close to, but no equal to, a

3. Of the six functions graphed in Figure 1, only the first three approach l at a.

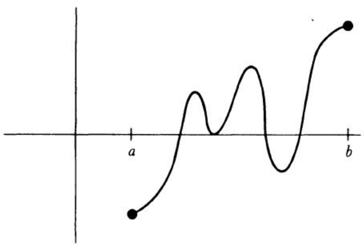


FIGURE 1

4. Notice that although $g(a)$ is not defined and $h(a)$ is defined "the wrong way", it is still true that g and h approach l near a.

5. This is because we explicitly ruled out, in our definition, the necessity of ever considering the value of the function at a

   - — it is only necessary that $f(x)$ should be close to l for x close to a, but unequal to a.

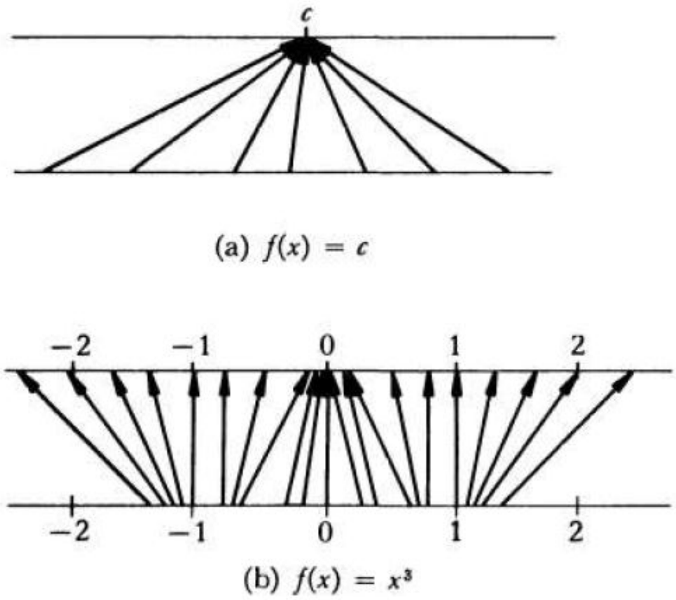
6. We are simply not interested in the value of $f(a)$, or even in the question of whether $f(a)$ is defined.

---

7. One convenient way of picturing the assertion that f approaches / near a is provided by a method of drawing functions that was not mentioned in Chapter 4.

8. In this method, we draw two straight lines, each representing $\mathbb{R}$, and arrows from a point x in one, to $f(x)$ in the other.

9. Figure 2 illustrates such a picture for two different functions.



(a) $f(x) = c$

(b) $f(x) = x^3$

FIGURE 2

10. Now consider a function f whose drawing looks like Figure 3.

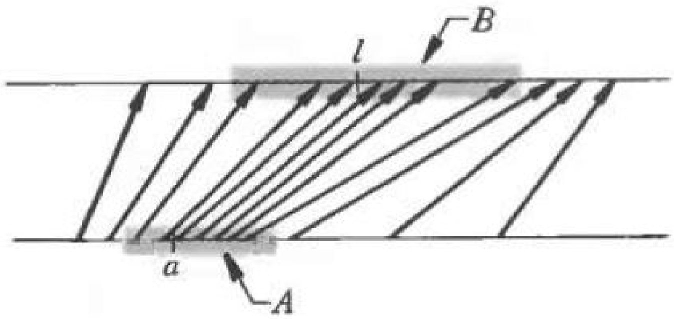


FIGURE 3

11. Suppose we ask that $f(x)$ be close to l, say within the open interval B which has been drawn in Figure 3.

12. This can be guaranteed if we consider only the numbers x in the interval A of Figure 3.

13. (In this diagram we have chosen the largest interval which will work; any smaller interval containing a could have been chosen instead.)

14. If we choose a smaller interval B' (Figure 4) we will, usually, have to choose a smaller A',

    - but no matter how small we choose the open interval B,

    there is always supposed to be some open interval A which works.

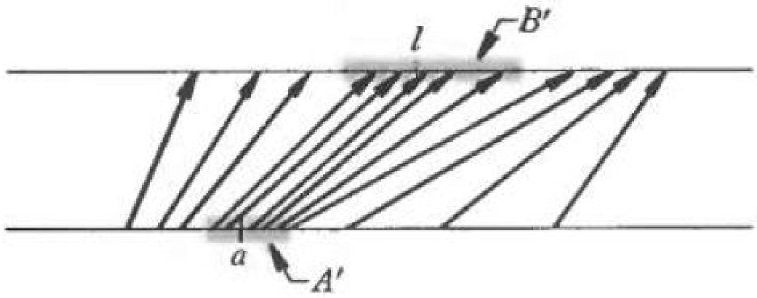


FIGURE 4

15. A similar pictorial interpretation is possible in terms of the graph of f, but in this case
    - the interval B must be drawn on the vertical axis, and
    - the set A on the horizontal axis.
16. The fact that $f(x)$ is in B when x is in A means that
    - the part of the graph lying over A is contained in the region which is bounded by the horizontal lines through the end points of B;
    - compare Figure 5(a), where a valid interval A has been chosen, with Figure 5(b), where A is too large.

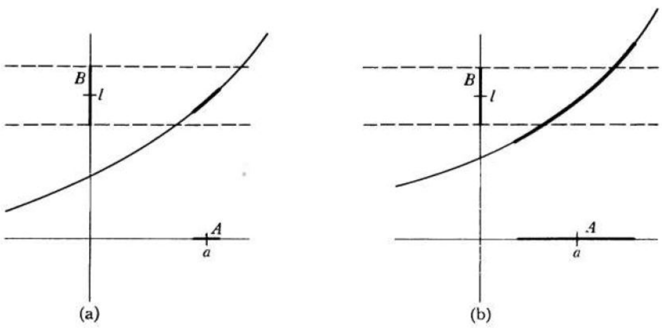


FIGURE 5

17. In order to apply our definition to a particular function, let us consider

$$f(x) = x \sin\left(\frac{1}{x}\right) \tag{5.1}$$

(Figure 6).

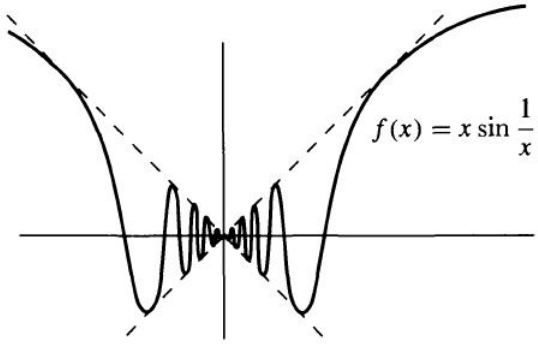


FIGURE 6

18. Despite the erratic behavior of this function near 0 it is clear, at least intuitively, that f approaches 0 near 0, and it is certainly to be hoped that our definition will allow us to reach the same conclusion.

19. In the case we are considering, both a and l of the definition are 0, so we must ask if we can get

$$f(x) = x \sin \frac{1}{x} \tag{5.2}$$

as close to 0 as desired if we require that x be sufficiently close to 0, but $\neq 0$.

20. To be specific, suppose we wish to get

$$x \sin \frac{1}{x} \text{ within } \frac{1}{10} \text{ of } 0. \tag{5.3}$$

21. This means we want

$$-\frac{1}{10} < x \sin \frac{1}{x} < \frac{1}{10} \tag{5.4}$$

or, more sufficiently,

$$\left| \sin \left(\frac{1}{x} \right) \right| \leq \frac{1}{10}. \tag{5.5}$$

22. Now this is easy.

23. Since

$$\left| \sin \frac{1}{x} \right| \leq 1, \forall x \neq 0 \tag{5.6}$$

we have

$$\left| x \sin \frac{1}{x} \right| \leq |x|, \forall x \neq 0 \tag{5.7}$$

24. This means that if

$$x < \frac{1}{10} \text{ and } x \neq 0, \tag{5.8}$$

then

$$\left| x \sin \frac{1}{x} \right| < \frac{1}{10}; \tag{5.9}$$

- in other words,

$$x \sin \frac{1}{x} \text{ is within } \frac{1}{10} \text{ of } 0 \tag{5.10}$$

- provided that x is within

$$\frac{1}{10} \text{ of } 0, \text{ but } \neq 0. \tag{5.11}$$

25. There is nothing special about the number $\frac{1}{10}$;

- it is just as easy to guarantee that

$$|f(x) - 0| < \frac{1}{100} \tag{5.12}$$

- simply require that

$$\frac{1}{100}, \text{ but } x \neq 0. \tag{5.13}$$

26. In fact, if we take any positive number ε we can make

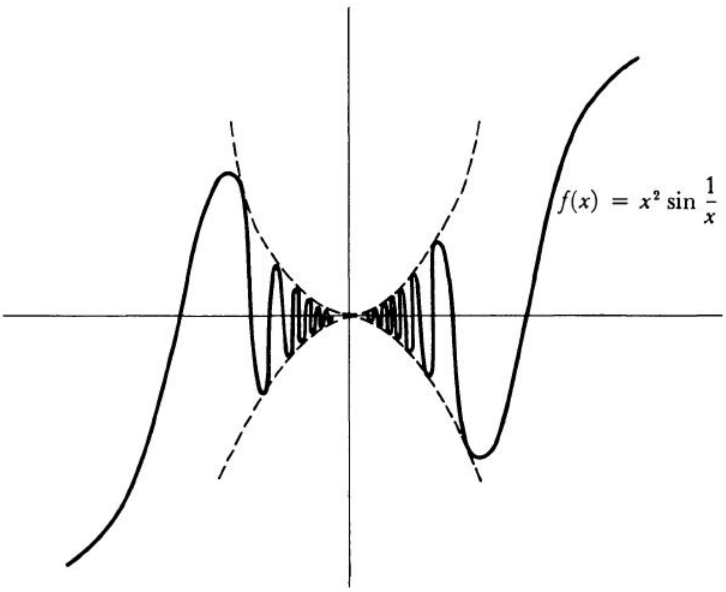
$$|f(x) - 0| < \varepsilon \tag{5.14}$$

simply by requiring that $|x| < \varepsilon$, and $x \neq 0$

27. For the function

$$f(x) = x^2 \sin \frac{1}{x}$$

(Figure 7) it seems even clearer that f approaches 0 near 0.



28. If, for example, we want

$$|x^2 \sin \frac{1}{x}| < \frac{1}{10} \tag{5.15}$$

then we certainly need only require that

$$|x| < \frac{1}{10} \text{ and } x \neq 0, \tag{5.16}$$

since this implies that $|x^2| < \dfrac{1}{100}$ and consequently

$$|x^2 \sin \frac{1}{x}| \leq |x^2| < \frac{1}{100} < \frac{1}{10} \tag{5.17}$$

29. (We could do even better, and allow

$$|x| < \frac{1}{\sqrt{10}} \text{ and } x \neq 0, \tag{5.18}$$

but there is no particular virtue in being as economical as possible.)

30. In general, if $\varepsilon > 0$, to ensure that

$$|x^2 \sin \frac{1}{x}| < \varepsilon \tag{5.19}$$

we need only require that

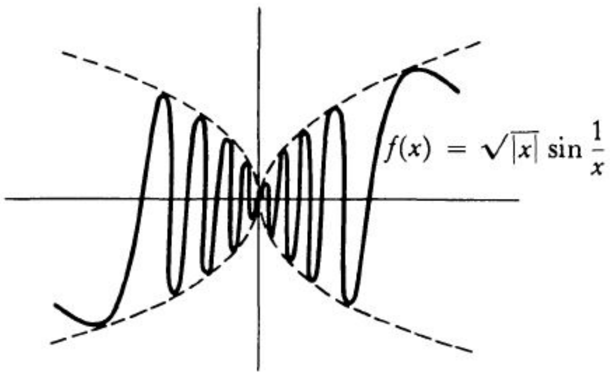
$$|x| < \varepsilon \text{ and } x \neq 0, \tag{5.20}$$

provided that $\varepsilon \leq 1$.

31. If we are given an ε which is greater than 1
    - (it might be, even thought it is "small" ε's which are of interest),
    - then it does not suffice to require that $|x| < \varepsilon$,
    - but it certainly suffices to require that $|x| < 1$ and $x \neq 0$.

32. As a third example, consider the function

$$f(x) = \sqrt{|x|} \sin\left(\frac{1}{x}\right) \text{ (Figure 8).} \tag{5.21}$$



33. In order to make

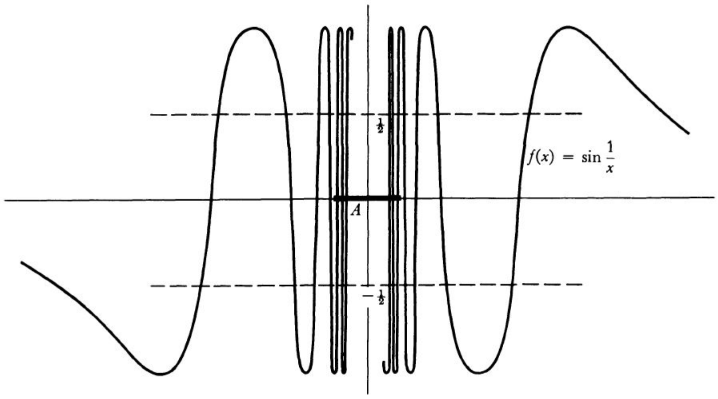
$$\left|\sqrt{|x|}\sin\frac{1}{x}\right| < \varepsilon \tag{5.22}$$

we can require that

$$|x| < \varepsilon^2 \text{ and } x \neq 0 \tag{5.23}$$

(the algebra is left to you).

34. Finally, let us consider the function $f(x) = \sin\frac{1}{x}$ (Figure 9).



35. For this function it is *false* that f approaches 0 near 0.

36. This amounts to saying that it is not true for every number $\varepsilon > 0$ that we can get

$$|f(x) - 0| < \varepsilon \tag{5.24}$$

by choosing x sufficiently small, and $\neq 0$.

37. To show this we simply have to find one $\varepsilon > 0$ for which the condition

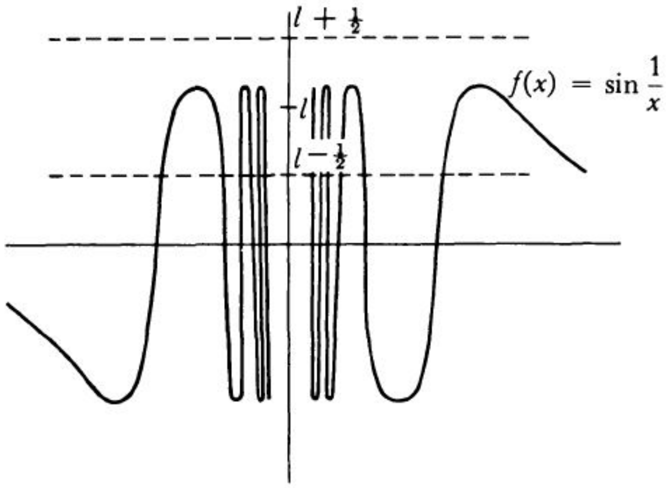
$$|f(x) - 0| < \varepsilon \tag{5.25}$$

cannot be guaranteed, no matter how small we require $|x|$ to be.

38. In fact, $\varepsilon = \frac{1}{2}$ will do:

- it is impossible to ensure that $|f(x)| < \frac{1}{2}$ no matter how small we require $|x|$ to be;
- for if A is any interval containing 0, there is some number $x = \frac{1}{(90 + 360n)}$ which is in this interval, and for this x we have $f(x) = 1$.

39. This same argument can be used (Figure 10) to show that f does not approach any number near 0.



40. To show this we must again find, for any particular number l, some number $\varepsilon > 0$ so that

$$|f(x) - l| < \varepsilon \tag{5.26}$$

is *not* true, no matter how small x is required to be.

41. The choice $\varepsilon = \dfrac{1}{2}$ works for any number l; that is, no matter how small we require $|x|$ to be, we cannot ensure that

$$|f(x) - l| < \frac{1}{2}. \tag{5.27}$$

42. The reason is, that for any interval A containing 0 there is some

$$x_1 = \frac{1}{(90 + 360n)} \tag{5.28}$$

in this interval, so that

$$f(x_1) = 1 \tag{5.29}$$

and also some

$$x_2 = \frac{1}{(270 + 360m)} \tag{5.30}$$

in this interval, so that

$$f(x_2) = -1 \tag{5.31}$$

43. But the interval from $l - \dfrac{1}{2}$ to $l + \dfrac{1}{2}$ cannot contain both –1 and 1, since its total length is only 1; so we cannot have

$$|1 - l| < \frac{1}{2} \text{ and also } |-1-l| < \frac{1}{2} \tag{5.32}$$

no mater what l is

44. The phenomenon exhibited by $f(x) = \sin\dfrac{1}{x}$ near 0 can occur in many ways.

45. If we consider the function

$$f(x) = \begin{cases} 0, \ x \text{ irrational} \\ 1, \ x \text{ rational}, \end{cases} \tag{5.33}$$

then no matter what a is, f does not approach any number l near a.

46. In fact, we cannot make

$$|f(x) - l| < \frac{1}{4} \tag{5.34}$$

no matter how close we bring x to a,

- because in any interval around a there are numbers x with $f(x) = 0$,
- and also numbers x with $f(x) = 1$, so that we would need

$$|0 - l| < \frac{1}{4} \text{ and also } |1 - l| < \frac{1}{4}. \tag{5.35}$$

47. An amusing variation on this behavior is presented by the function shown in Figure 11:

$$f(x) = \begin{cases} x, & x \text{ rational} \\ 1, & x \text{ irrational}, \end{cases} \tag{5.36}$$

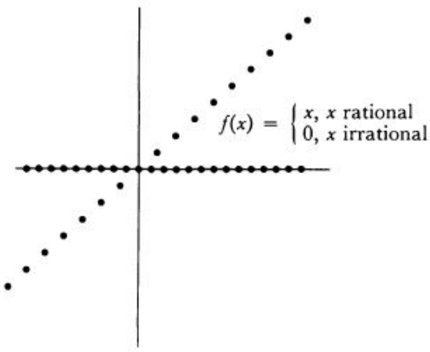


FIGURE 11

48. The behavior of this function is "opposite" to that of

$$g(x) = \sin\frac{1}{x} \tag{5.37}$$

it approaches 0 at 0, but does not approach any number at a, if $a \neq 0$.

49. By now you should have no difficulty convincing yourself that this is true.

50. As a contrast to the functions considered so far, which have been quite pathological, we will now examine some of the simplest functions.

51. If $f(x) = c$, then f approaches c near a, for every number a.

52. In fact, to ensure that

$$|f(x) - c| < \varepsilon \tag{5.38}$$

one does not need to restrict x to be near a at all; the condition is automatically satisfied (Figure 12).

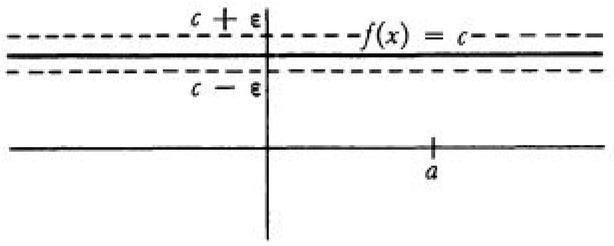


FIGURE 12

53. As a slight variation, let f be the function shown in Figure 13:

$$f(x) = \begin{cases} -1, & x < 0 \\ 1\ , & x > 0 \end{cases} \tag{5.39}$$

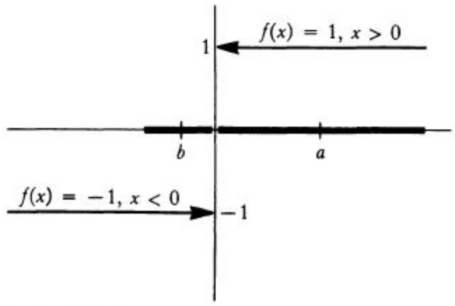


FIGURE 13

54. If $a > 0$, then f approaches l near a: indeed, to ensure that

$$|f(x) - l| < \varepsilon \tag{5.40}$$

it certainly suffices to require that

$$|x - a| < a, \tag{5.41}$$

since this implies

$$-a < x - a \tag{5.42}$$

$$\text{or } 0 < x \tag{5.43}$$

so that $f(x) = 1$.

55. Similarly, if $b < 0$, then f approaches -1 near b: to ensure that

$$|f(x)–(-1)| < \varepsilon \tag{5.44}$$

it suffices to require that $|x - b| < -b$.

56. Finally, as you may easily check, f does not approach any number near 0.

57. The function $f(x) = x$ is easily dealt with.

58. Clearly f approaches a near a: to ensure that

$$|f(x) - a| < \varepsilon \tag{5.45}$$

we just have to require that

$$|x–a| < \varepsilon. \tag{5.46}$$

59. The function $f(x) = x^2$ requires a little more work.

60. To show that f approaches a^2 near a , we must decide how to ensure that

$$|x^2 - a^2| < \varepsilon. \tag{5.47}$$

61. Factoring looks like the most promising procedure: we want

$$|x - a| \cdot |x + a| < \varepsilon \tag{5.48}$$

62. Obviously the factor $|x + a|$ is the one that will cause trouble.

63. On the other hand, there is no need to make $|x + a|$ particularly small;

   - as long as we know some bound on the values of $|x + a|$ we will be in good shape.

64. For example, if

$$|x + a| < 1,000,000, \tag{5.49}$$

then we will just need to require that

$$|x - a| < \frac{\varepsilon}{10000000}. \tag{5.50}$$

65. Therefore, to begin with, let us require that $|x - a| < 1$

   - (any positive number other than I would do just as well);
   - presumably this will ensure that x is not too large, and consequently that $|x + a|$ is not too large.

66. As a matter of fact, Problem 1-12 shows that

$$|x| - |a| \leq |x - a| < 1, \tag{5.51}$$

so

$$|x| < 1 + |a|, \tag{5.52}$$

and consequently

$$|x + a| \leq |x| + |a| < 2|a| + 1 \tag{5.53}$$

.

67. Now we need only the additional requirement that

$$|x - a| < \frac{\varepsilon}{(2|a| + 1)}. \tag{5.54}$$

68. In other words, if

$$|x - a| < \min\left(1, \frac{\varepsilon}{2|a| + 1}\right), \tag{5.55}$$

then

$$|x^2 - a^2| < \varepsilon. \tag{5.56}$$

69. Naturally,

$$\min\left(1, \frac{\varepsilon}{(2|a| + 1)}\right) \tag{5.57}$$

will just be

$$\frac{\varepsilon}{(2|a| + 1)} \tag{5.58}$$

for small ε.

70. Precisely the same sort of trick will show that if

$$f(x) = x^3, \tag{5.59}$$

then f approaches a^3 near a.

71. In fact, if

$$|x - a| < \min\left(1, \frac{\varepsilon}{(1 + |a|)^2 + |a|(1 + |a|) + |a|^2}\right), \tag{5.60}$$

then

$$|x^3 - a^3| < \varepsilon. \tag{5.61}$$

72. The proof of this assertion will show where the weird denominator comes from:

73. If $|x - a| < 1$, then $|x| < |a| + 1$, and consequently

$$\begin{aligned} |x^2 + ax + a^2| &\le |x|^2 + |a| \cdot |x| + |a|^2 \\ &< (1 + |a|)^2 + |a|(1 + |a|) + |a|^2 \end{aligned} \tag{5.62}$$

74. Therefore

$$\begin{aligned} |x^3 - a^3| &= |x - a| \cdot |x^2 + ax + a^2| \\ &< \frac{\varepsilon}{(1 + |a|)^2 + |a|(1 + |a|) + |a|^2} \cdot \left[(1 + |a|)^2 + |a|(1 + |a|) + |a|^2\right] \\ &= \varepsilon \end{aligned} \tag{5.63}$$

75. The time has now come to point out that of the many demonstrations about limits which we have given, not one has been a real proof.

76. The fault lies not with our reasoning, but with our definition.

77. If our provisional definition of a function was open to criticism, our provisional definition of approaching a limit is even more vulnerable.

78. This definition is simply not sufficiently precise to be used in proofs.

79. It is hardly clear how one "makes" $f(x)$ close to 1

    - (whatever "close" means)

    by "requiring" x to be sufficiently close to a

    - (however close "sufficiently" close is supposed to be).

80. Despite the criticisms of our definition you may feel

    - (I certainly hope you do)

    that our arguments were nevertheless quite convincing.

81. In order to present any sort of argument at all, we have been practically forced to invent the real definition.

82. It is possible to arrive at this definition in several steps, each one clarifying some obscure phrase which still remains.

83. Let us begin, once again, with the provisional definition:

    - The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to, a.

84. The very first change which we made in this definition was to note that making $f(x)$ close to l meant making

$$|f(x) - l| \tag{5.64}$$

    small, and similarly for x and a:

    - The function f approaches the limit l near a, if we can make $|f(x) - l|$ as small as we like by requiring that $|x - a|$ be sufficiently small, and $x \neq a$.

85. The second, more crucial, change was to note that making

$$|f(x) - l| \tag{5.65}$$

    "as small as we like" means making

$$|f(x) - l| < \varepsilon \tag{5.66}$$

    for any $\varepsilon > 0$ that happens to be given us:

    - The function f approaches the limit l near a, if for every number $\varepsilon > 0$ we can make

$$|f(x) - l| < \varepsilon \tag{5.67}$$

      by requiring that $|x - a|$ be sufficiently small, and $x \neq a$.

86. There is a common pattern to all the demonstrations about limits which we have given.

87. For each number $\varepsilon > 0$ we found some other positive number, δ say, with the property that if

$$x \neq a \text{ and } |x-a| < \delta, \tag{5.68}$$

then

$$|f(x) - l| < \varepsilon. \tag{5.69}$$

88. For the function

$$f(x) = x \sin\left(\frac{1}{x}\right) \tag{5.70}$$

- (with $a = 0$, $l = 0$), the number δ was just the number ε;
- for

$$f(x) = \sqrt{|x|} \sin\left(\frac{1}{x}\right), \tag{5.71}$$

it was ε^2 for

$$f(x) = x^2 \tag{5.72}$$

it was the minimum of

$$1 \text{ and } \frac{\varepsilon}{2|a|+1}. \tag{5.73}$$

89. In general, it may not be at all clear how to find the number δ, given ε, but it is the condition

$$|x-a| < \delta \tag{5.74}$$

which expresses how small "sufficiently" small must be:

- The function f approaches the limit l near a, if for every $\varepsilon > 0$
- there is some $\delta > 0$ such that, for all x, if $|x-a| < \delta$ and $x \neq a$, then $|f(x) - 1| < \varepsilon$.

90. This is practically the definition we will adopt.

91. We will make only one trivial change, nothing that "$|x-a| < \delta$ and $x \neq a$" can just as well be expressed "$0 < |x-a| < \delta$".

DEFINITION

The function f approaches the limit l near a means:

- for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if

$$0 < |x-a| < \delta, \tag{5.75}$$

then

$$|f(x) - l| < \varepsilon. \tag{5.76}$$

92. This definition is so important

- (everything we do from now on depends on it) that proceeding any further without knowing it is hopeless.

93. If necessary memorize it, like a poem!

94. That, at least, is better than stating it incorrectly;

    - if you do this you are doomed to give incorrect proofs.

95. A good exercise in giving correct proofs is to review every fact already demonstrated about functions approaching limits, giving real proofs of each.

96. This requires writing down the correct definition of what you are proving, but not much more-all the algebraic work has been done already.

97. When proving that f does not approach l at a, be sure to negate the definition correctly:

98. If it is not true that

    - for every $\varepsilon > 0$
    - there is some $\delta > 0$ such that, for all x if

    $$0 < |x - a| < \delta, \tag{5.77}$$

    then

    $$|f(x) - l| < \varepsilon \tag{5.78}$$

    - then there is some $\varepsilon > 0$ such that for every $\delta > 0$
    - there is some x which satisfies

    $$0 < |x - a| < \delta \tag{5.79}$$

    but not

    $$|f(x) - l| < \varepsilon \tag{5.80}$$

99. Thus, to show that the function

    $$f(x) = \sin\frac{1}{x} \tag{5.81}$$

    does not approach 0 near 0, we consider $\varepsilon = \dfrac{1}{2}$ and note that for every $\varepsilon > 0$ there is some x with

    $$0 < |x - 0| < \delta \tag{5.82}$$

    but not

    $$\left|\sin\frac{1}{x} - 0\right| < \frac{1}{2} \tag{5.83}$$

    -namely, an x of the form

    $$\frac{1}{(90 + 360n)}, \tag{5.84}$$

    where n is so large that

    $$\frac{1}{(90 + 360n)} < \delta. \tag{5.85}$$

100. As an illustration of the use of the definition of a function approaching a limit, we have reserved the function shown in Figure 14, a standard example, but one of the most complicated:

$$f(x) = \begin{cases} 0\ , \ x \text{ irrational}, \ 0 < x < 1 \\ \dfrac{1}{q}, \ \ x = \dfrac{p}{q} \text{ in lowest terms}, \ 0 < x < 1 \end{cases} \tag{5.86}$$

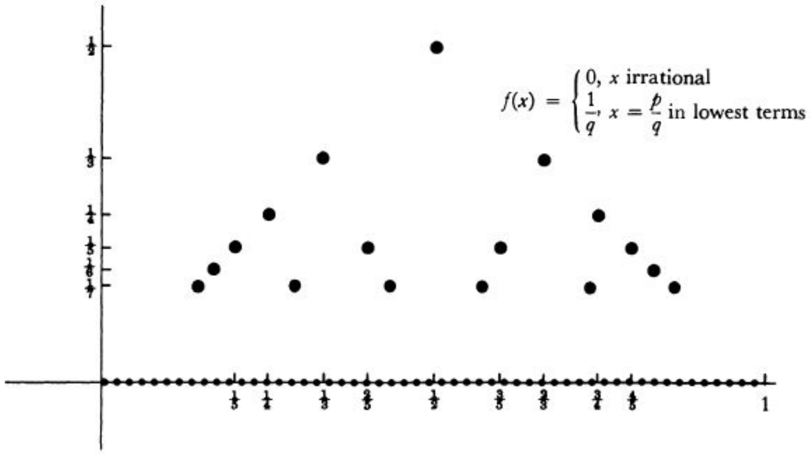


FIGURE 14

101. (Recall that $\frac{p}{q}$ is in lowest terms if p and q are integers with no common factor and $q > 0$.)
For any number a, with $0 < a < 1$, the function f approaches 0 at a.

102. To prove this, consider any number $\varepsilon > 0$.

103. Let n be a natural number so large that $\frac{1}{n} \leq \varepsilon$.

104. Notice that the only numbers x for which $|f(x) - 0| < \varepsilon$ could be false are:

$$\frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; ...; \frac{1}{n}, ..., \frac{n-1}{n} \tag{5.87}$$

105. (If a is rational, then a might be one of these numbers.)

106. However many of these numbers there may be, there are, at any rate, only finitely many.

107. Therefore, of all these numbers, one is closest to a; that is,

$$\left| \frac{p}{q} - a \right| \tag{5.88}$$

is smallest for one $\frac{p}{q}$ among these numbers.

108. (If a happens to be one of these numbers, then consider only the values $\left|\frac{p}{q} - a\right|$ for $\frac{p}{q} \neq a$.)

109. This closest distance may be chosen as the δ.

110. For if

$$0 < |x - a| < \delta, \tag{5.89}$$

then x is not one of

$$\frac{1}{2}, ..., \frac{n-1}{n} \tag{5.90}$$

and therefore $|f(x) - 0| < \varepsilon$ is true.

111. This completes the proof.

112. Note that our description of the δ which works for a given ε is completely adequate—there is no reason why we must give a formula for δ in terms of ε.

113. Armed with our definition, we are now prepared to prove our first theorem;

   - you have probably assumed the result all along, which is a very reasonable thing to do.

114. This theorem is really a test case for our definition:

   - if the theorem could not be proved, our definition would be useless.

THEOREM 1

A function cannot approach two different limit near a.

- In other words, if f approaches l near a, and f approaches m near a, then $l = m$.

PROOF

115. Since this is our first theorem about limits it will certainly be necessary to translate the hypotheses according to the definition.

116. Since f approaches l near a, we know that for any $\varepsilon > 0$ there is some number $\delta_1 > 0$ such that, for all x, if

$$0 < |x - a| < \delta_1, \tag{5.91}$$

then

$$|f(x) - 1| < \varepsilon. \tag{5.92}$$

117. We also know, since f approaches m near a, that there is some $\delta_2 > 0$ such that, for all x, if

$$0 < |x - a| < \delta_2, \tag{5.93}$$

then

$$|f(x) - m| < \varepsilon. \tag{5.94}$$

118. We have had to use two numbers, δ_1 and δ_2, since there is no guarantee that the δ which works in one definition will work in the other.

119. But, in fact, it is now easy to conclude that for any $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if

$$0 < |x - a| < \delta, \tag{5.95}$$

then

$$|f(x) - l| < \varepsilon \text{ and } |f(x) - m| < \varepsilon; \tag{5.96}$$

we simply choose

$$\delta = \min(\delta_1, \delta_2). \tag{5.97}$$

120. To complete the proof we just have to pick a particular $\varepsilon > 0$ for which the two conditions

$$|f(x) - l| < \varepsilon \text{ and } |f(x) - m| < \varepsilon \tag{5.98}$$

cannot both hold, if $l \neq m$.

121. The proper choice is suggested by Figure 15.

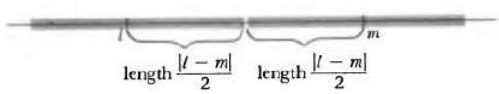


FIGURE 15

122. If $l \neq m$, so that

$$|l - m| > 0, \tag{5.99}$$

we can choose $\dfrac{|l-m|}{2}$ as our ε.

123. It follows that there is a $\delta > 0$ such that, for all x, if

$$0 < |x - a| < \delta, \tag{5.100}$$

then

$$|f(x) - l| < \frac{|l-m|}{2} \text{ and } |f(x) - m| < \frac{|l-m|}{2}. \tag{5.101}$$

124. This implies that for $0 < |x - a| < \delta$ we have

$$\begin{aligned} |l - m| &= |l - f(x) + f(x) - m| \\ &\leq |l - f(x)| + |f(x) - m| \\ &< \frac{|l-m|}{2} + \frac{|l-m|}{2} \\ &= |l - m| \end{aligned} \tag{5.102}$$

a contradiction. ■

125. The number l which f approaches near a is denoted by $\lim\limits_{x \to a} f(x)$

- (read: the limit of $f(x)$ as x approaches a).

126. This definition is possible only because of Theorem 1, which ensures that $\lim_{x\to a} f(x)$ never has to stand for two different numbers.

127. The equation

$$\lim_{x\to a} f(x) = l \tag{5.103}$$

has exactly the same meaning as the phrase

$$f \text{ approaches } l \text{ near } a. \tag{5.104}$$

128. The possibility still remains that f does not approach l near a, for any l, so that $\lim_{x\to a} f(x) = l$ is false for every number l.

129. This is usually expressed by saying that "$\lim_{x\to a} f(x)$ does not exist."

130. Notice that our new notation introduces an extra, utterly irrelevant letter x,

- which could be replaced by t, y, or any other letter which does not already appear — the symbols

$$\lim_{x\to a} f(x), \qquad \lim_{t\to a} f(t), \qquad \lim_{y\to a} f(y), \tag{5.105}$$

- all denote precisely the same number, which depends on f and a, and has nothing to do with x, t, or y.

- (these letters, in fact, do not denote anything at all).

131. A more logical symbol would be something like $\lim_{a} f$, but this notation, despite its brevity, is so infuriatingly rigid that almost no one has seriously tried to use it.

132. The notation $\lim_{x\to a} f(x)$ is much more useful because a function f often has no simple name, even though it might be possible to express $f(x)$ by a simple formula involving x.

133. Thus, the short symbol

$$\lim_{x\to a} (x^2 + \sin x) \tag{5.106}$$

could be paraphrased only by the standard symbolism is illustrated by the expressions

$$\begin{aligned} &\lim_{x\to a} x + t^3 \\ &\lim_{t\to a} x + t^3 \end{aligned} \tag{5.107}$$

134. The first means the number which f approaches near a when

$$f(x) = x + t^3, \quad \text{for all } x \tag{5.108}$$

the second mean the number which f approaches near a when

$$f(t) = x + t^3, \quad \text{for all } t \tag{5.109}$$

135. You should have little difficult (especially if you consult Theorem 2) proving that

$$\begin{aligned} \lim_{x\to a} x + t^3 &= a + t^3 \\ \lim_{t\to a} x + t^3 &= x + a^3 \end{aligned} \tag{5.110}$$

136. These examples illustrate the main advantage of our notation, which is its flexibility.

137. In fact, the notation $\lim_{x\to a} f(x)$ is so flexible that there is some danger of forgetting what it really means.

138. Here is a simple exercise in the use of this notation, which will be important later: first interpret precisely, and then prove the equality of the expressions

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{h\to 0} f(a+h) \tag{5.111}$$

139. An important part of this chapter is the proof of a theorem which will make it easy to find many limits.

140. The proof depends upon certain properties of inequalities and absolute values, hardly surprising when one considers the definition of limit.

141.
- Although these facts have already been stated in Problems 1-20, 1-21, and 1-22, because of their importance they will be presented once again,
- in the form of a lemma (a lemma is an auxiliary theorem, a result that justifies its existence only by virtue of its prominent role in the proof of another theorem).

142. The lemma says, roughly, that if x is close to x_0, and y is close to y_0, then $x+y$ will be close to x_0+y_0, and xy will be close to x_0y_0, and $\frac{1}{y}$ will be close to $\frac{1}{y_0}$.

143. This intuitive statement is much easier to remember than the precise estimates of the lemma,
- and it is not unreasonable to read the proof of Theorem 2 first,
- in order to see just how these estimates are used.

LEMMA

(a) If

$$|x - x_0| < \frac{\varepsilon}{2} \text{ and } |y - y_0| < \frac{\varepsilon}{2} \tag{5.112}$$

then

$$|(x+y) - (x_0+y_0)| < \varepsilon \tag{5.113}$$

(b) If

$$|x - x_0| < \min\left(1, \frac{\varepsilon}{2(|y_0|+1)}\right) \text{ and } |y - y_0| < \frac{\varepsilon}{2(|x_0|+1)}, \tag{5.114}$$

then

$$|xy - x_0y_0| < \varepsilon. \tag{5.115}$$

(c) If

$$y_0 \neq 0 \text{ and } |y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right), \tag{5.116}$$

then

$$y \neq 0 \text{ and } \left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon. \tag{5.117}$$

PROOF

(a)

$$\begin{aligned} |(x+y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\leq |x - x_0| + |y - y_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned} \tag{5.118}$$

(b) 1. Since $|x - x_0| < 1$ we have

$$|x| - |x_0| \leq |x - x_0| < 1, \tag{5.119}$$

so that

$$|x| < 1 + |x_0|. \tag{5.120}$$

2. Thus

$$\begin{aligned} |xy - x_0y_0| &= |x(y - y_0) + y_0(x - x_0)| \\ &\leq |x| \cdot |y - y_0| + |y_0| \cdot |x - x_0| \\ &< (1 + |x_0|) \cdot \frac{\varepsilon}{2(|x_0| + 1)} + |y_0| \cdot \frac{\varepsilon}{2(|y_0| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned} \tag{5.121}$$

(c) 1. We have

$$|y_0| - |y| \leq |y - y_0| < \frac{y_0}{2}, \tag{5.122}$$

so $|y| > \dfrac{|y_0|}{2}$.

2. In particular, $y \neq 0$,and

$$\frac{1}{|y|} < \frac{2}{|y_0|}. \tag{5.123}$$

3. Thus

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \frac{|y_0 - y|}{|y| \cdot |y_0|} < \frac{2}{|y_0|} \cdot \frac{1}{|y_0|} \cdot \frac{\varepsilon|y_0|^2}{2} = \varepsilon. \blacksquare \tag{5.124}$$

THEOREM 2

1. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

$$\begin{aligned} &(1) \quad \lim_{x\to a}(f+g)(x) = l+m; \\ &(2) \quad \lim_{x\to a}(f\cdot g)(x) = l\cdot m. \end{aligned} \tag{5.125}$$

2. Moreover, if $m \neq 0$, then

$$(3) \quad \lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{m}. \tag{5.126}$$

PROOF:

1. The hypothesis means that for every $\varepsilon > 0$ there are $\delta_1, \delta_2 > 0$ such that, for all x, if

$$0 < |x-a| < \delta_1, \tag{5.127}$$

then

$$|f(x) - l| < \varepsilon \tag{5.128}$$

and if

$$0 < |x-a| < \delta_2 \tag{5.129}$$

then

$$|g(x) - m| < \varepsilon \tag{5.130}$$

2. This means (since, after all, $\frac{\varepsilon}{2}$ is also a positive number) that there are $\delta_1, \delta_2 > 0$ such that, for all x, if

$$0 < |x-a| < \delta_1 \tag{5.131}$$

then

$$|f(x) - l| < \frac{\varepsilon}{2} \tag{5.132}$$

and if

$$0 < |x-a| < \delta_2, \tag{5.133}$$

then

$$|g(x) - m| < \frac{\varepsilon}{2}. \tag{5.134}$$

3. Now let

$$\delta = \min(\delta_1, \delta_2). \tag{5.135}$$

4. If

$$0 < |x-a| < \delta, \tag{5.136}$$

then

$$0 < |x-a| < \delta_1 \tag{5.137}$$

and

$$0 < |x-a| < \delta - 2 \tag{5.138}$$

are both true, so both

$$|f(x) - l| < \frac{\varepsilon}{2} \text{ and } |g(x) - m| < \frac{\varepsilon}{2}. \tag{5.139}$$

are true.

5. But by part (1) of the lemma this implies that

$$|(f+g)(x)-(l-m)|<\varepsilon. \tag{5.140}$$

6. This proves (1).
7. To prove (2) we proceed similarly, after consulting part (2) of the lemma, $\varepsilon>0$
   - there are $\delta_1,\delta_2>0$ such that, for all x,

   if

$$0<|x-a|<\delta_1, \tag{5.141}$$

   then

$$|f(x)-l|<\min\left(1,\frac{\varepsilon}{2(|m|+1)}\right), \tag{5.142}$$

   and if

$$0<|x-a|<\delta_2, \tag{5.143}$$

   then

$$|g(x)-m|<\frac{\varepsilon}{2(|l|+1)}. \tag{5.144}$$

8. Again let

$$\delta=\min(\delta_1,\delta_2). \tag{5.145}$$

9. If $0<|x-a|<\delta$, then

$$|f(x)-l|<\min\left(1,\frac{\varepsilon}{2(|m|+1)}\right) \tag{5.146}$$

   and

$$|g(x)-m|<\frac{\varepsilon}{2(|l|+1)}. \tag{5.147}$$

10. So, by the lemma,

$$|(f\cdot g)(x)-l\cdot m|<\varepsilon, \tag{5.148}$$

   and this proves (2).
11. Finally, if $\varepsilon>0$ there is a $\delta>0$ such that, for all x, if

$$0<|x-a|<\delta, \tag{5.149}$$

   then

$$|g(x)-m|<\min\left(\frac{|m|}{2},\frac{\varepsilon|m|^2}{2}\right). \tag{5.150}$$

144. But according to part (3) of the lemma this means, first, that $g(x)\neq 0$, so $\frac{1}{g}(x)$ makes sense, and second that

$$|\frac{1}{g}(x)-\frac{1}{m}|<\varepsilon. \tag{5.151}$$

145. This proves (3).

THEOREM 2

1. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

$$\begin{aligned} &(1) \quad \lim_{x\to a}(f+g)(x) = l+m; \\ &(2) \quad \lim_{x\to a}(f\cdot g)(x) = l\cdot m. \end{aligned} \tag{5.152}$$

2. Moreover, if $m \neq 0$, then

$$(3) \quad \lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{m}. \tag{5.153}$$

146. Using Theorem 2 we can prove, trivially, such facts as

$$\lim_{x\to a}\frac{x^3+7x^5}{x^2+1} = \frac{a^3+7a^5}{a^2+1}, \tag{5.154}$$

without going through the laborious process of finding a δ, given an ε.

147. We must begin with

$$\lim_{x\to a} 7 = 7, \tag{5.155}$$

$$\lim_{x\to a} 1 = 1, \tag{5.156}$$

$$\lim_{x\to a} x = a, \tag{5.157}$$

but these are easy to prove directly.

148. If we want to find the δ, however, the proof of Theorem 2 amounts to a prescription for doing this.

149. Suppose, to take a simpler example, that we *want* to find a δ such that, for all x, if

$$0 < |x-a| < \delta \tag{5.158}$$

then

$$|x^2+x-(a^2+a)| < \varepsilon. \tag{5.159}$$

THEOREM 2

1. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

$$\begin{aligned} &(1) \quad \lim_{x\to a}(f+g)(x) = l+m; \\ &(2) \quad \lim_{x\to a}(f\cdot g)(x) = l\cdot m \end{aligned} \tag{5.160}$$

2. Moreover, if $m \neq 0$, then

$$(3) \quad \lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{m} \tag{5.161}$$

150. Consulting the proof of Theorem 2(1), we see that we must first find δ_1 and $\delta_2 > 0$ such that, for all x, if

$$0 < |x - a| < \delta_1, \tag{5.162}$$

then

$$|x^2 - a^2| < \frac{\varepsilon}{2} \tag{5.163}$$

if

$$0 < |x - a| < \delta_2, \tag{5.164}$$

then

$$|x - a| < \frac{\varepsilon}{2}. \tag{5.165}$$

151. Since we have already given proofs that $\lim_{x \to a} x^2 = a^2$ and $\lim_{x \to a} x = a$, we know how to do this:

$$\delta_1 = \min\left(1, \frac{\frac{\varepsilon}{2}}{2|a|+1}\right), \ \delta_2 = \frac{\varepsilon}{2} \tag{5.166}$$

152. Thus we can take

$$\begin{aligned} \delta &= \min\ (\delta_1, \delta_2) \\ &= \min\left(\min\left(1, \frac{\frac{\varepsilon}{2}}{2|a|+1}\right), \frac{\varepsilon}{2}\right). \end{aligned} \tag{5.167}$$

153. If a $\neq 0$, the same method can be used to find a $\delta > 0$ such that, for all x, if

$$0 < |x - a| < \delta, \tag{5.168}$$

then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon. \tag{5.169}$$

THEOREM 2

1. If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

$$\begin{aligned} &(1) \quad \lim_{x \to a} (f + g)(x) = l + m; \\ &(2) \quad \lim_{x \to a} (f \cdot g)(x) = l \cdot m. \end{aligned} \tag{5.170}$$

2. Moreover, if $m \neq 0$, then

$$(3) \quad \lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}. \tag{5.171}$$

154. The proof of Theorem 2(3) shows that the second condition will follow if we find a $\delta > 0$ such that, for all x, if

$$0 < |x - a| < \delta, \tag{5.172}$$

$$|x^2 - a^2| < \min\left(\frac{|a|^2}{2}, \frac{\varepsilon |a|^4}{2}\right). \tag{5.173}$$

155. Thus we can take

$$\delta = \min\left(1, \frac{\min(\frac{|a|^2}{2}, \frac{\varepsilon|a|^4}{2})}{2|a|+1}\right) \tag{5.174}$$

156. Naturally, these complicated expressions for δ can be simplified considerably, after they have been derived.

THEOREM 2

1. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

$$\begin{aligned} &(1) \quad \lim_{x\to a}(f+g)(x) = l+m; \\ &(2) \quad \lim_{x\to a}(f\cdot g)(x) = l\cdot m. \end{aligned} \tag{5.175}$$

2. Moreover, if $m \neq 0$, then

$$(3) \quad \lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{m}. \tag{5.176}$$

157. One technical detail in the proof of Theorem 2 deserves some discussion.

158. In order for $\lim_{x\to a} f(x)$ to be defined it is, as we know, not necessary for f to be defined at a, nor is it necessary for f to be defined at all points $x \neq a$.

159. However, there must be some $\delta > 0$ such that $f(x)$ is defined for x satisfying

$$0 < |x-a| < \delta. \tag{5.177}$$

160. Otherwise the clause "if

$$0 < |x-a| < \delta, \tag{5.178}$$

then

$$|f(x) - l| < \varepsilon\text{"} \tag{5.179}$$

would make no sense at all, since the symbol $f(x)$ would make no sense for some x's.

161. If f and g are two functions for which the definition makes sense, it is easy to see that the same is true for $f+g$ and $f\cdot g$

162. But this is not so clear for $\frac{1}{g}$, since $\frac{1}{g}$ is undefined for x with $g(x) = 0$.

163. However, this fact gets established in the proof of Theorem 2(3).

164. There are times when we would like to speak of the limit which f approaches at a, even though there is no $\delta > 0$ such that $f(x)$ is defined for x satisfying

$$0 < |x-a| < \delta. \tag{5.180}$$

165. For example, we want to distinguish the behavior of the two functions shown in Figure 16, even though they are not defined for numbers less than a.

166. For the function of Figure 16(a) we write

$$\lim_{x\to a^+} f(x) = l \text{ or } \lim_{x\downarrow a^-} f(x) = l. \quad (5.181)$$

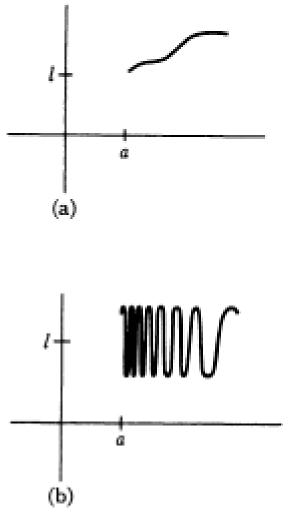


FIGURE 16

167. (The symbols on the left are read: the limit of $f(x)$ as x approaches a from above.)

168. These "limits from above" are obviously closely related to ordinary limits, and the definition is very similar:

- $\lim_{x\to a^+} f(x) = l$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if

$$0 < x - a < \delta, \quad (5.182)$$

then

$$|f(x) - l| < \varepsilon. \quad (5.183)$$

169. (The condition " $0 < x - a < \delta$" is equivalent to "$0 < |x - a| < \delta$ and $x > a$".)

170. "Limits from below" (Figure 17) are defined similarly:

- $\lim_{x\to a^+} f(x) = l$
- (or $\lim_{x\uparrow a^+} f(x) = l$) means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if

$$0 < a - x < \delta, \quad (5.184)$$

then

$$|f(x) - l| < \varepsilon. \quad (5.185)$$

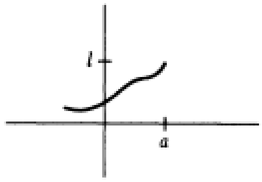


FIGURE 17

171. It is quite possible to consider limits from above and below even if f is defined for numbers both greater and less than a.

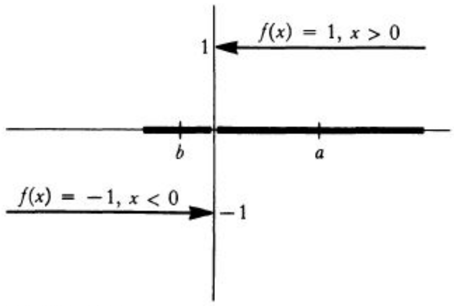


FIGURE 13

172. Thus, for the function f of Figure 13, we have

$$\lim_{x \to 0^+} f(x) = 1 \text{ and } \lim_{x \to 0^-} f(x) = -1. \tag{5.186}$$

173. It is an easy exercise (Problem 29) to show that

- $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal.

174. Like the definitions of limits from above and below, which have been smuggled into the text informally, there are other modifications of the limit concept which will be found useful.

175. In Chapter 4 it was claimed that if x is large, then $\sin \frac{1}{x}$ is close to 0.

176. This assertion is usually written

$$\lim_{x \to \infty} \sin \frac{1}{x} = 0 \tag{5.187}$$

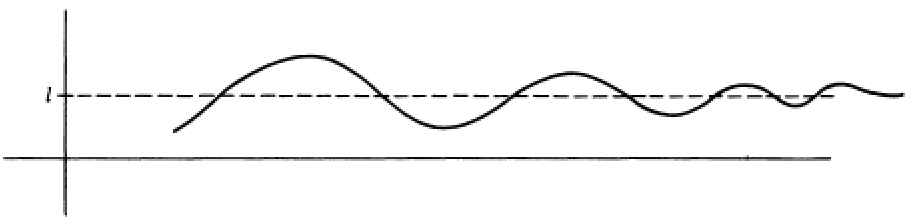


FIGURE 18

177. The symbol $\lim_{x\to\infty} f(x)$ is read "the limit of $f(x)$ as x approaches ∞ ," or "as x becomes infinite," and a limit of the form $\lim_{x\to\infty} f(x)$ is often called a limit at infinity.

178. Figure 18 illustrates a general situation where $\lim_{x\to\infty} f(x) = l$.

179. Formally, $\lim_{x\to\infty} f(x) = l$ means that for every $\varepsilon > 0$ there is a number N such that, for all x, if

$$x > N, \tag{5.188}$$

then

$$|f(x) - l| < \varepsilon. \tag{5.189}$$

180. The analogy with the definition of ordinary limits should be clear:
    - whereas the condition "$0 < |x - a| < \delta$" expresses the fact that x is close to a,
    - the condition "$x > N$" expresses the fact that x is large.

181. We have spent so little time on limits from above and below, and at infinity,
    - because the general philosophy behind the definitions should be clear if you understand the definition of ordinary limits
    - (which are by far the most important).

182. Many exercises on these definitions are provided in the Problems, which also contain several other types of limits which are occasionally useful.

5.2 Problem

1. Find the following limits.
    - (These limits all follow, after some algebraic manipulations, from the various parts of Theorem 2;
    - be sure you know which ones are used in each case, but don't bother listing them.)

    (a) $\lim_{x\to 1} \dfrac{x^2 - 1}{x + 1}$.

    (b) $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$.

    (c) $\lim_{x\to 3} \dfrac{x^3 - 8}{x - 2}$.

    (d) $\lim_{x\to y} \dfrac{x^n - y^n}{x - y}$.

    (e) $\lim_{y\to x} \dfrac{x^n - y^n}{x - y}$.

    (f) $\lim_{h\to 0} \dfrac{\sqrt{a+h} - \sqrt{a}}{h}$.

2. Find the following limits.

(a) $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$.

(b) $\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x}$.

(c) $\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

3. In each of the following cases, find a δ such that $|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

   (a) $f(x) = x^4; l = a^4$.

   (b) $f(x) = \frac{1}{x}; a = 1, l = 1$.

   (c) $f(x) = x^4 + \frac{1}{x}; a = 1, l = 2$.

   (d) $f(x) = \frac{x}{1 + sin^2 x}; a = 0, l = 0$.

   (e) $f(x) = \sqrt{|x|}; a = 0, l = 0$.

   (f) $f(x) = \sqrt{x}; a = 1, l = 1$.

4. For each of the functions in Problem 4-17, decide for which numbers a the limit $\lim_{x \to a} f(x)$ exists.

5. (a) Do the same for each of the functions in Problem 4-19.

   (b) Same problem, if we use infinite decimals ending in a string of O's instead of those ending in a string of 9's.

6. Suppose the functions f and g have the following property: for all $\varepsilon > 0$ and all x,

$$\text{if } 0 < |x - 2| < \sin^2\left(\frac{\varepsilon^2}{9}\right) + \varepsilon, \text{ then } |f(x) - 2| < \varepsilon \tag{5.190}$$

$$\text{if } 0 < |x - 2| < \varepsilon^2, \text{ then } |g(x) - 4| < \varepsilon \tag{5.191}$$

   - For each $\varepsilon > 0$ find a $\delta > 0$ such that, for all x,

   (a) if $0 < |x - 2| < \delta$, then $|f(x) + g(x) - 6| < \varepsilon$

   (b) if $0 < |x - 2| < \delta$, then $|f(x)g(x) - 8| < \varepsilon$

   (c) if $0 < |x - 2| < \delta$, then $\left|\frac{1}{g(x)} - \frac{1}{4}\right| < \varepsilon$

   (d) if $0 < |x - 2| < \delta$, then $\left|\frac{f(x)}{g(x)} - \frac{1}{2}\right| < \varepsilon$

7. Give an example of a function f for which the following assertion is $false$:

   - If $|f(x) - l| < \varepsilon$ when $0 < |x - a| < \delta$, then $|f(x) - l| < \frac{\varepsilon}{2}$ when $0 < |x - a| < \frac{\delta}{2}$.

8. (a) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, can $\lim_{x \to a} [f(x) + g(x)]$ or $\lim_{x \to a} f(x) \cdot g(x)$ exist?

(b) If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} [f(x)+g(x)]$ exist, must $\lim_{x\to a} g(x)$ exist?

(c) If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a} [f(x)+g(x)]$ exist?

(d) If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} f(x)g(x)$ exist, does it follow that $\lim_{x\to a} g(x)$ exists?

9. Prove that $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$.

   - (This is mainly an exercise in understanding what the terms mean.)

10. (a) Prove that $\lim_{x\to a} f(x) = l$ if and only if $\lim_{x\to a} [f(x) - l] = 0$.
    - First see why the assertion is obvious; then provide a rigorous proof.
    - In this chapter most problems which ask for proofs should be treated in the same way.)

    (b) Prove that $\lim_{x\to 0} f(x) = \lim_{x\to a} f(x-a)$.

    (c) Prove that $\lim_{x\to 0} f(x) = \lim_{x\to 0} f(x^3)$.

    (d) Give an example where $\lim_{x\to 0} f(x^2)$ exists, but $\lim_{x\to 0} f(x)$ does not.

11. - Suppose there is a $\delta > 0$ such that $f(x) = g(x)$ when $0 < |x-a| < \delta$.
    - Prove that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.
    - In other words, $\lim_{x\to a} f(x)$ depends only on the values of $f(x)$ for x near a.
    - This fact is often expressed by saying that limits are a "local property".
    - (It will clearly help to use δ', or some other letter, instead of δ, in the definition of limits.

12. (a) Suppose that $f(x) = g(x)$ for all x.
    - Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided that these limits exist.

    (b) How can the hypotheses by weakened?

    (c) If $f(x) < g(x)$ for all x, does it necessarily follow that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$

13. Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$.

    - Prove that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$, and exist $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$ (Draw a picture!)

14. (a) Prove that if and $\lim_{x\to a} \frac{f(x)}{x} = l$ and $b \neq 0$,then $\lim_{x\to a} \frac{f(bx)}{x} = bl$.
    - Hint: write $\frac{f(x)}{x} = b[\frac{f(bx)}{bx}]$.

    (b) What happens if b = 0?

    (c) Part (a) enable us to find $\lim_{x\to 0} \frac{\sin(2x)}{x}$ in terms of $\lim_{x\to a} \frac{\sin(x)}{x}$.

    (d) Find this limit in another way.

15. Evaluate the following limits in terms of the number $\alpha = \lim_{x\to 0} \frac{\sin(x)}{x}$

    (a) $\lim_{x\to 0} \frac{\sin(2x)}{x}$.

(b) $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$.

(c) $\lim_{x\to 0} \frac{\sin^2(2x)}{x}$.

(d) $\lim_{x\to 0} \frac{\sin^2(2x)}{x^2}$.

(e) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$.

(f) $\lim_{x\to 0} \frac{\tan^2(x)+2x}{x+x^2}$.

(g) $\lim_{x\to 0} \frac{x\sin(x)}{1-\cos(x)}$.

(h) $\lim_{x\to 0} \frac{\sin(x+h)-\sin(x)}{h}$.

(i) $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$.

(j) $\lim_{x\to 0} \frac{x^2(3+\sin(x))}{1-\cos(x)}$.

(k) $\lim_{x\to 1} (x^2-1)^3 \sin(\frac{1}{x-1})^3$.

16. (a) Prove that if $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} |f|(x) = |l|$.

(b) Prove that if $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then $\lim_{x\to a} \max(f,g)(x) = \max(l,m)$ and similarly for min.

17. (a) Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist, i.e., show that $\lim_{x\to 0} \frac{1}{x} = l$ is false for every number l.

(b) Prove that $\lim_{x\to 1} \frac{1}{(x-1)}$ does not exist.

18.
- Prove that if $\lim_{x\to a} f(x) = l$, then there is a number $\delta > 0$ and a number M such that $|f(x)| < M$ if $0 < |x-a| < \delta$.
- What does this mean pictorially?
- Hint: Why does it suffice to prove that $l-1 < f(x) < l+1$ for $0 < |x-a| < \delta$?

19. Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x, then $\lim_{x\to a} f(x)$ does not exist for any a.

20. Prove that if $f(x) = x$ for rational x, and $f(x) = -x$ for irrational x, then $\lim_{x\to a} f(x)$ does not exist if $a \neq 0$.

21. (a) Prove that if $\lim_{x\to 0} g(x) = 0$, then $\lim_{x\to 0} g(x)\sin\left(\frac{1}{x}\right) = 0$.

(b)
- Generalize this fact as follows: If $\lim_{x\to 0} g(x) = 0$ and $|h(x)| \leq M$ for all x, then $\lim_{x\to 0} g(x)h(x) = 0$.
- (Naturally it is unnecessary to do part (a) if you succeed in doing part (b);)

- actually the statement of part (b) may make it easier than (a)–that's one of the values of generalization.

22.
- Consider a function f with the following property: if g is any function for which $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0}[g(x)+f(x)]$ also does not exist.
- Prove that this happens if and only if $\lim_{x\to 0} f(x)$ does exist.
- Hint: This is actually very easy: the assumption that $\lim_{x\to 0} f(x)$ does not exist leads to an immediate contradiction if you consider the right g.

23.
- This problem is the analogue of Problem 22 when $f+g$ is replaced by $f\cdot g$.
- In this case the situation is considerably more complex, and the analysis requires several steps (those in search of an especially challenging problem can attempt an independent solution).
- Suppose that $\lim_{x\to 0} f(x)$ exists and is $\neq 0$.Prove that if $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0} f(x)g(x)$ also does not exist.
- Prove the same result if $\lim_{x\to 0}|f(x)|=\infty$.(The precise definition of this sort of limit is given in Problem 37.
-
  - Prove that if neither of these two conditions holds, then there is a function g such that $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0} f(x)g(x)$ does exist.
  - Hint: Consider separately the following two cases:
    1. for some $\varepsilon>0$ we have $|f(x)|<\varepsilon$ for all sufficiently small x.
    2. For every $\varepsilon>0$, there are arbitrarily small x with $|f(x)|<\varepsilon$. In the second case, begin by choosing points x_n with $|x_n|<\frac{1}{n}$ and $|f(x_n)<\frac{1}{n}|$.

24. Suppose that A_n is, for each natural number n, some finite set of numbers in [0, 1], and that A_n and A_m have no members in common if $m\neq n$.

- Define f as follow:

$$f(x)=\begin{cases}\frac{1}{n}, x \text{ in } A_n\\ 0, x \text{ not in } A_n \text{ for any } n.\end{cases} \tag{5.192}$$

prove that $\lim_{x\to a} f(x)=0$ for all a in [0,1].

25. Explain why the following definitions of $\lim_{x\to a} f(x)=l$ are all correct:

- For every $\delta>0$ there is an $\varepsilon>0$ such that, for all x,

(a) if $0<|x-a|<\varepsilon$, then $|f(x)-l|<\delta$.
(b) if $0<|x-a|<\varepsilon$, then $|f(x)-l|\leq\delta$.
(c) if $0<|x-a|<\varepsilon$, then $|f(x)-l|<5\delta$.
(d) if $0<|x-a|<\frac{\varepsilon}{10}$, then $|f(x)-l|<\delta$.

26. Give examples to show that the following definitions of $\lim_{x\to a} f(x)=l$ are not correct.

(a) For all $\delta>0$ there is an $\varepsilon>0$ such that if $0<|x-a|<\delta$, then $|f(x)-l|<\varepsilon$.

(b) For all $\varepsilon > 0$ there is a $\delta > 0$ such that if $|f(x) - l| < \varepsilon$, then $0 < |x - a| < \delta$.

27. For each of the functions in Problem 4-17 indicate for which numbers a the one-sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist.

28. (a) Do the same for each of the functions in Problem 4-19.
    (b) Also consider what happens if decimals ending in O's are used instead of decimals ending in 9's.

29. Prove that $\lim_{x \to a} f(x)$ exist if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

30. Prove that

    (a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(-x)$.
    (b) $\lim_{x \to 0} f(|x|) = \lim_{x \to 0^+} f(x)$.
    (c) $\lim_{x \to 0} f(x^2) = \lim_{x \to 0^+} f(x)$.

    - (These equations, and others like them, are open to several interpretations.
    - They might mean only that the two limits are equal if they both exist;
    - or that if a certain one of the limits exists, the other also exists and is equal to it;
    - or that if either limit exists, then the other exists and is equal to it.
    - Decide for yourself which interpretations are suitable.)

31.
    - Suppose that $\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$.
    - (Draw a picture to illustrate this assertion.)
    - Prove that there is some $\delta > 0$ such that $f(x) < f(y)$ whenever $x < a < y$ and $|x - a| < \delta$ and $|y - a| < \delta$.
    - Is the converse true?

32.
    - Prove that $\lim_{x \to \infty} \frac{(a_n x^n + \cdots + a_0)}{(b_m x^m + \cdots + a_0)}$ exists if and only if $m \geq n$.
    - What is the limit when $m = n$? When $m > n$?
    - Hint: the one easy limit is $\lim_{x \to \infty} \frac{1}{x^k} = 0$;
    - do some algebra so that this is the only information you need.

33. Find the following limits.

    (a) $\lim_{x \to \infty} \frac{x + \sin^3(x)}{5x + 6}$.
    (b) $\lim_{x \to \infty} \frac{x + \sin(x)}{x^2 + 5}$.
    (c) $\lim_{x \to \infty} \sqrt{x^2 + x} - x$.
    (d) $\lim_{x \to \infty} \frac{x^2(1 + \sin^2(x))}{(x + \sin(x))^2}$.

34. Prove that $\lim_{x\to 0^+} f(\frac{1}{x}) = \lim_{x\to\infty} f(x)$

35. Find the following limits in terms of the number $\alpha = \lim_{x\to 0} \frac{\sin(x)}{x}$

(a) $\lim_{x\to\infty} \frac{\sin(x)}{x}$.

(b) $\lim_{x\to\infty} x\sin(\frac{1}{x})$.

36. Define "$\lim_{x\to-\infty} f(x) = l$."

(a) Find $\lim_{x\to-\infty} \frac{(a_n x^n + \cdots + a_0)}{(b_m x^m + \cdots + a_0)}$

(b) Prove that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(-x)$.

(c) Prove that $\lim_{x\to 0^-} f(\frac{1}{x}) = \lim_{x\to-\infty} f(x)$.

37.
- We define $\lim_{x\to a} f(x) = \infty$ to mean that for all N
- there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $f(x) > N$.
- (Draw an appropriate picture!)

(a) Show that $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$.

(b) Prove that if $f(x) > \varepsilon > 0$ for all x, and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{|g(x)|} = \infty. \quad (5.193)$$

38. (a) Define $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, and $\lim_{x\to a} f(x) = \infty$.
- (Or at least convince yourself that you could write down the definitions if you had the energy.
- How many other such symbols can you define?)

(b) Prove that $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

(c) Prove that $\lim_{x\to 0^+} f(x) = \infty$ if and only if $\lim_{x\to\infty} f(\frac{1}{x}) = \infty$.

39. Find the following limits, when they exist.

(a) $\lim_{x\to\infty} \frac{x^3 + 4x - 7}{7x^2 - x + 1}$

(b) $\lim_{x\to\infty} x(1 + \sin^2(x))$

(c) $\lim_{x\to\infty} x\sin^2(x)$

(d) $\lim_{x\to\infty} x^2 \sin(\frac{1}{x})$

(e) $\lim_{x\to\infty} \sqrt{x^2 + 2x} - x$

(f) $\lim_{x\to\infty} x(\sqrt{x+2}-\sqrt{x})$

(g) $\lim_{x\to\infty} \frac{\sqrt{|x|}}{x}$

40. (a) Find the perimeter of a regular n-gon inscribed in a circle of radius r; use radian measure for any trigonometric functions involved.[Answer: $2rn\sin(\frac{\pi}{n})$.]

    (b) What value does this perimeter approach as n becomes very large?

41.
- After sending the manuscript for the first edition of this book off to the printer,
- I thought of a much simpler way to prove that $\lim_{x\to a} x^2 = a^2$ and $\lim_{x\to a} x^3 = a^3$, without going through all the factoring tricks on page 95.
- Suppose, for example, that we want to prove that $\lim_{x\to a} x^2 = a^2$, where $a > 0$.
- Given $\varepsilon > 0$, we simply let δ be the minimum of

$$\sqrt{a^2+\varepsilon}-a \tag{5.194}$$

and

$$a-\sqrt{a^2-\varepsilon} \tag{5.195}$$

(see Figure 19); then

$$|x-a|<\delta \tag{5.196}$$

implies that

$$\sqrt{a^2-\varepsilon}<x<\sqrt{a^2+\varepsilon}, \tag{5.197}$$

so

$$a^2-\varepsilon<x^2<a^2+\varepsilon, \tag{5.198}$$

or

$$|x^2-a^2|<\varepsilon. \tag{5.199}$$

- It is fortunate that these pages had already been set, so that I couldn't make these changes, because this “ Proof ” is completely fallacious.
- Wherein lies the fallacy?

6 Continuous Function (p.113)

6.1 Context

1. If f is an arbitrary function, it is not necessarily true that

$$\lim_{x\to a} f(x) = f(a) \tag{6.1}$$

2. In fact, there are many ways this can fail to be true.

3. For example, f might not even be defined at a, in which case the equation makes no sense (Figure 1).

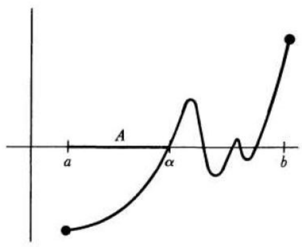


FIGURE 1

4. Again, $\lim_{x\to a} f(x)$ might not exist (Figure 2).

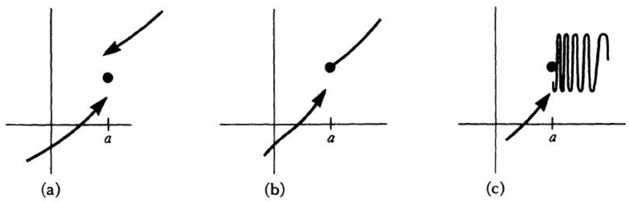


FIGURE 2

5. Finally, as illustrated in Figure 3, even if f is defined at a and $\lim_{x\to a} f(x)$ exists, the limit might not equal $f(a)$.

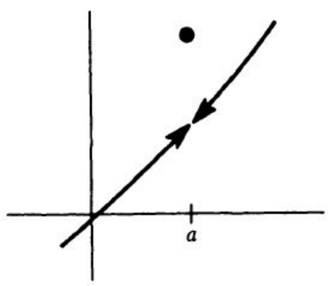


FIGURE 3

6. We would like to regard all behavior of this type as abnormal and honor, with some complimentary designation, functions which do not exhibit such peculiarities.

7. The term which has been adopted is "continuous".
8. Intuitively, a function f is continuous if the graph contains no breaks, jumps, or wild oscillations.
9. Although this description will usually enable you to decide whether a function is continuous simply by looking at its graph (a skill well worth cultivating)
   - it is easy to be fooled, and the precise definition is *very* important.

DEFINITION
The function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a). \tag{6.2}$$

10. We will have no difficulty finding many examples of functions which are, or are not, continuous at some number a
    - — every example involving limits provides an example about continuity, and Chapter 5 certainly provides enough of these.
11. The function $f(x) = \sin\left(\frac{1}{x}\right)$ is not continuous at 0, because it is not even defined at 0, and the same is true of the function $g(x) = x \sin\left(\frac{1}{x}\right)$.
12. On the other hand, if we are willing to extend the second of these functions, that is, if we wish to define a new function G by

$$G(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ a & x = 0, \end{cases} \tag{6.3}$$

13. then the choice of $a = G(0)$ can be made in such a way that G will be continuous at 0
    - — to do this we can (if fact, we must) define $G(0) = 0$ (Figure 4).

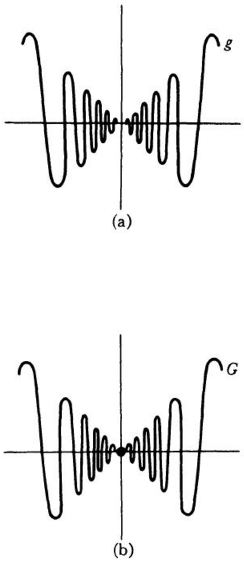


FIGURE 4

14. This sort of extension is not possible for f;

   - if we define

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ a & x = 0, \end{cases} \tag{6.4}$$

   then f will not be continuous at 0, no matter what a is, because $\lim_{x\to a} f(x)$ does not exist.

15. The function

$$f(x) = \begin{cases} x & \text{x rational} \\ 0 & \text{x irrational}, \end{cases} \tag{6.5}$$

is not continuous at a, if $a \neq 0$, since $\lim_{x\to a} f(x)$ does not exist.

16. However, $\lim_{x\to a} f(x) = 0 = f(0)$, so f is continuous at precisely one point 0.

17. The functions $f(x) = c$, $g(x) = x$, and $h(x) = x^2$ are continuous at all numbers a, since

$$\lim_{x\to a} f(x) = \lim_{x\to a} c = c = f(a), \tag{6.6}$$

$$\lim_{x\to a} g(x) = \lim_{x\to a} x = a = g(a), \tag{6.7}$$

$$\lim_{x\to a} h(x) = \lim_{x\to a} x^2 = a^2 = h(a), \tag{6.8}$$

18. Finally, consider the function

$$f(x) = \begin{cases} 0 & \text{x irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lower terms.} \end{cases} \tag{6.9}$$

19. In Chapter 5 we showed that $\lim_{x\to a} f(x) = 0$ for all a.

20. Since $0 = f(a)$ only when a is irrational, this function is continuous at a if a is irrational, but not if a is rational.

21. It is even easier to give examples of continuity if we prove two simple theorems.

> THEOREM 1
> If f and g are continuous at a, then
>
> 1. $f + g$ is continuous at a.
> 2. $f \cdot g$ is continuous at a.
> 3. Moreover, if $g(a) \neq 0$, then $\frac{1}{g}$ is continuous at a.

Proof:

22. Since f and g are continuous at a,

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a) \tag{6.10}$$

23. By Theorem 2(1) of Chapter 5 this implies that

$$\lim_{x \to a} f(x) + g(x) = f(a) + g(a) = (f + g)(a), \tag{6.11}$$

which is just the assertion that $f + g$ is continuous at a.

24. The proofs of parts (2) and (3) are left to you.■

25. Starting with the functions $f(x) = c$ and $f(x) = x$, which are continuous at a, for every a, we can use Theorem 1 to conclude that a function

$$f(x) = \frac{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}{c_m x^m + c_{m-1} x^{m-1} + \cdots + c_0} \tag{6.12}$$

is continuous at every point in its domain.

26. But it is harder to get much further than that.

27. When we discuss the sine function in detail it will be easy to prove that sin is continuous at a for all a; let us assume this fact meanwhile.

28. A function like

$$f(x) = \frac{\sin^2(x) + x^2 + x^4 \sin(x)}{\sin^{27}(x) + 4x^2 \sin^2(x)} \tag{6.13}$$

can now be proved continuous at every point in its domain.

29. But we are still unable to prove the continuity of a function like $f(x) = \sin(x^2)$;
    - we obviously need a theorem about the composition of continuous functions.

30. Before stating this theorem, the following point about the composition of continuous functions.

31. If we translate the equation $\lim_{x \to a} f(x) = f(a)$ according to the definition of limits, we obtain for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, if

$$0 < |x - a| < \delta, \tag{6.14}$$

then

$$|f(x) - f(a)| < \varepsilon. \tag{6.15}$$

32. But in this case, where the limit is $f(a)$, the phrase

$$0 < |x - a| < \delta \tag{6.16}$$

may be change to the simpler condition

$$|x - a| < \delta. \tag{6.17}$$

33. Since if $x = a$ it is certainly true that

$$|f(x) - f(a)| < \varepsilon. \tag{6.18}$$

THEOREM 2
If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. (Notice that f is required to be continuous at $g(a)$, not at a.)

Proof.

a. Let $\varepsilon > 0$.
b. We wish to find a $\delta > 0$ such that for all x.
c. If $|x - a| < \delta$, then $|(f \circ g)(x) - (f \circ g)(a)| < \varepsilon$.
d. i.e., $|f(g(x)) - f(g(a))| < \varepsilon$

□

34. We first use continuity of f to estimate how close $g(x)$ must be to $g(a)$ in order for this inequality to hold.
35. Since f is continuous at $g(a)$, there is a $\delta' > 0$ such that for all y, if

$$|y - g(a)| < \delta', \tag{6.19}$$

then

$$|f(y) - f(g(a))| < \varepsilon \tag{6.20}$$

36. In particular, this mean that if

$$|g(x) - g(a)| < \delta', \tag{6.21}$$

then

$$|f(g(x)) - f(g(a))| < \varepsilon. \tag{6.22}$$

37. We now use continuity of g to estimate how close x must be to a in order for the inequality

$$|g(x) - g(a)| < \delta \tag{6.23}$$

to hold.

38. The number δ is a positive number just like any other positive number;

   - we can therefore take δ' as the $\varepsilon(!)$ in the definition of continuity of g at a.

39. We conclude that there is a $\delta > 0$ such that, for all x, if

$$|x - a| < \delta, \tag{6.24}$$

then

$$|g(x) - g(a)| < \delta'. \tag{6.25}$$

40. Combining (2.19) and (2.20) we see that for all x, if

$$|x - a| < \delta, \tag{6.26}$$

then

$$|f(g(x)) - f(g(a))| < \varepsilon \tag{6.27}$$

41. We can now reconsider the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0, \end{cases} \tag{6.28}$$

42. We have already noted that f is continuous at 0.

43. A few applications of Theorems 1 and 2, together with the continuity of sin, show that f is also continuous at a, for $a \neq 0$.

44. Functions like

$$f(x) = \sin(x^2 + \sin(x + \sin^2(x^3))) \tag{6.29}$$

should be equally easy for you to analyze.

45.
   - The few theorems of this chapter have all been related to continuity of functions at a single point,
   - but the concept of continuity doesn't begin to be really interesting
   - until we focus our attention on functions which are continuous at all points of some interval.

46. If f is continuous at x for all x in (a, b), then f is called continuous on (a, b).

47. Continuity on a closed interval must be defined a little differently; a function f is called continuous on $[a, b]$ if

   1. f is continuous at x for all x in (a, b),
   2. $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

48. Functions which are continuous on an interval are usually regarded as especially well behaved;

    - indeed continuity might be specified as the first condition which a “reasonable” function ought to satisfy.

49. A continuous function is sometimes described, intuitively, as one whose graph can be drawn without lifting your pencil from the paper.

50. Consideration of the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0, \end{cases} \tag{6.30}$$

    shows that this description is a little too optimistic,

    - but it is nevertheless true that there are many important results involving functions which are continuous on an interval.

51. There theorems are generally much harder than the ones in this chapter, but there is a simple theorem which forms a bridge between the two kinds of results.

52. The hypothesis of this theorem requires continuity at only a single point, but the conclusion describes the behavior of the function on some interval containing the point.

53. Although this theorem is really a lemma for later arguments, it is included here as a preview of things to come.

THEOREM 3

1. Suppose f is continuous at a, and $f(a) > 0$.
2. Then there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$.
3. Similarly, if $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $|x - a| < \delta$.

Proof. 1. Consider the case $f(a) > 0$.

2. Since f is continuous at a, if $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if

$$|x - a| < \delta, \tag{6.31}$$

   then

$$|f(x) - f(a)| < \varepsilon. \tag{6.32}$$

3. Since $f(a) > 0$ we can take $f(a)$ as the ε.
4. Thus there is $\delta > 0$ so that for all x, if

$$|x - a| < \delta, \tag{6.33}$$

   then

$$|f(x) - f(a)| < f(a), \tag{6.34}$$

   and this last inequality implies $f(x) > 0$.

5. A similar proof can be given in the case $f(a) < 0$;
   - take $\varepsilon = -f(a)$.
6. Or one can apply the first case to the function -f.

□

6.2 Problem

1. For which of the following functions f is there a continuous function F with domain $\mathbb{R}$ such that $F(x) = f(x)$ for all x in the domain of f
   (a) $f(x) = \dfrac{x^2 - 4}{x - 2}$
   (b) $f(x) = \dfrac{|x|}{x}$
   (c) $f(x) = 0$, x irrational.
   (d) $f(x) = \dfrac{1}{q}, x = \dfrac{p}{q}$ rational in lowest terms.
2. At which points are the functions of Problems 4-17 and 4-19 continuous?
3. (a) Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x.
   - Show that f is continuous at 0. (Notice that $f(0)$ must equal 0.)

   (b) Give an example of such a function f which is not continuous at any $a \neq 0$.
   (c) Suppose that g is continuous at 0 and $g(0) = 0$, and $|f(x)| \leq |g(x)|$.
   - Prove that f is continuous at 0.
4. Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.
5. For each number a, find a function which is continuous at a, but not at any other points.
6. (a) Find a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ but continuous at all other points.
   (b) Find a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, and at 0, but continuous at all other points.
7. Suppose that f satisfies $f(x + y) = f(x) + f(y)$, and that f is continuous at 0.
   - Prove that f is continuous at a for all a.
8. Suppose that f is continuous at a and $f(a) = 0$.
   - Prove that if $a \neq 0$, then $f + \alpha$ is nonzero in some open interval containing a.
9. (a) Suppose f is not continuous at a.
   - Prove that for some number $\varepsilon > 0$ there are numbers x arbitrarily close to a with $f(x) - f(a) > \varepsilon$.
   - Illustrate graphically.

   (b) Conclude that for some number $\varepsilon > 0$ either there are numbers x arbitrarily close to a with $f(x) < f(a) - \varepsilon$ or there are numbers x arbitrarily close to a with $f(x) > f(a) + \varepsilon$.

10. (a) Prove that if f is continuous at a, then so is $|f|$.
    (b) Prove that every continuous f can be written $f = E + O$, where E is even and continuous and O is odd and continuous.
    (c) Prove that if f and g are continuous, then so are $\max(f, g)$ and $\min(f, g)$
    (d) Prove that every continuous f can be written $f = g - h$, where g and h are non-negative and continuous.

11. Prove Theorem 1(3) by using Theorem 2 and continuity of the function $f(x) = \dfrac{1}{x}$.

12. (a) Prove that f is continuous at l and $\lim_{x\to a} g(x) = l$, then $\lim_{x\to a} f(g(x)) = f(l)$.
    - (You can go right back to the definitions,
    - but it is easier to consider the function G with $G(x) = g(x)$ for $x \neq a$, and $G(a) = l$.)

    (b) Show that if continuity of f at l is not assumed, then it is not generally true that

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)). \tag{6.35}$$

    - Hint: Try $f(x) = 0$ for $x \neq l$, and $f(l) = 1$.

13. (a) Prove that if f is continuous on $[a, b]$, then there is a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all x in $[a, b]$.
    - Hint: Since you obviously have a great deal of choice, try making g constant in $(-\infty, a]$ and $[b, \infty)$.

    (b) Give an example to show that this assertion is false if $[a, b]$ is replaced by (a, b).

14. (a)
    - Suppose that g and h are continuous at a, and that $g(a) = h(a)$.
    - Define $f(x)$ to be $g(x)$ if $x \geq a$ and $h(x)$ if $x \leq a$.
    - Prove that f is continuous at a.

    (b)
    - Suppose g is continuous on $[a, b]$ and h us continuous in $[b, c]$ and $g(b) = h(b)$.
    - Let $f(x)$ be $g(x)$ for x in $[a, b]$ and $h(x)$ for x in $[b, c]$.
    - Show that f is continuous on $[a, c]$.
    - (Thus, continuous functions can be "pasted together"".)

15. (a)
    - Prove that following version of Theorem 3 for "right-hand continuity"":
    - Suppose that $\lim_{x\to a^+} f(x) = f(a)$, and $f(a) > 0$.
    - Then there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $0 \leq x - a < \delta$.
    - Similarly, if $f(x) < 0$ for all x satisfying $0 \leq x - a < \delta$.

    (b) Prove a version of Theorem 3 when $\lim_{x\to b^-} f(x) = f(b)$.

16. if $\lim_{x\to a} f(x)$ exists, but is $\neq f(a)$, then f is said to have a **removable dis-continuity** at a.

    (a)
    - If $f(x) = \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 1$, does f have a removable discontinuity at 0?
    - What if $f(x) = x \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$, and $f(0) = 1$?

    (b)
    - Suppose f has a removable discontinuity at a.

- Let $g(x) = f(x)$ for $x \neq a$, and let $g(a) = \lim_{x \to a} f(x)$.
- Prove that g is continuous at a.
- (Don't work very hard; this is quite easy.)

(c)
- Let $f(x) = 0$ if x is irrational, and let $f\left(\frac{p}{q}\right) = \frac{1}{q}$ if $\frac{p}{q}$ is in the lowest terms.
- What is the function g defined by $g(x) = \lim_{y \to x} f(y)$?

(d)
- Let f be a function with the property that every point of discontinuity is a removable discontinuity.
- This means that $\lim_{y \to x} f(y))$ exists for all x, but f may be discontinuous at some (even infinitely many) number x.
- Define $g(x) = \lim_{y \to x} f(y)$.
- Prove that g is continuous.
- (This is not quite so easy as part(b).)

(e)
- Is there a function which is discontinuous at every point, and which has only removable discontinuities?
- (It is worth thinking about this problem now, but mainly as a test of intuition;
- even if you suspect the correct answer, you will almost certainly be unable to prove it at the present time.
- See Problem 22-33.)

17. Now that we have discovered the fallacy, it is almost obvious what additional property of the real numbers we need.
18. All we must do is say it properly and use it.
19. That is the business of the next chapter.

7 Three Hard Theorems (p.120)

7.1 Context

1. This chapter is devoted to three theorems about continuous functions, and some of their consequences.

2. The proofs of the three theorems themselves will not be given until the next chapter, for reasons which are explained at the end of this chapter.

THEOREM 1
If f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \tag{7.1}$$

then there is some x in $[a, b]$ such that

$$f(x) = 0. \tag{7.2}$$

3. (Geometrically, this means that the graph of a continuous function which starts below the horizontal axis and ends above it must cross this axis at some point, as in Figure 1.)

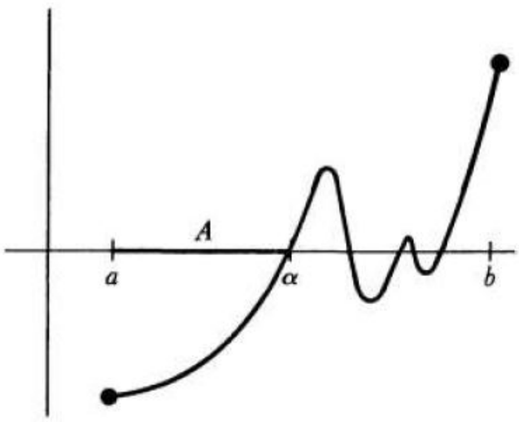


FIGURE 1

THEOREM 2
If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that

$$f(x) <= N \tag{7.3}$$

for all x in $[a, b]$.

4. (Geometrically, this theorem means that the graph of f lies below some line parallel to the horizontal axis, as in Figure 2.)

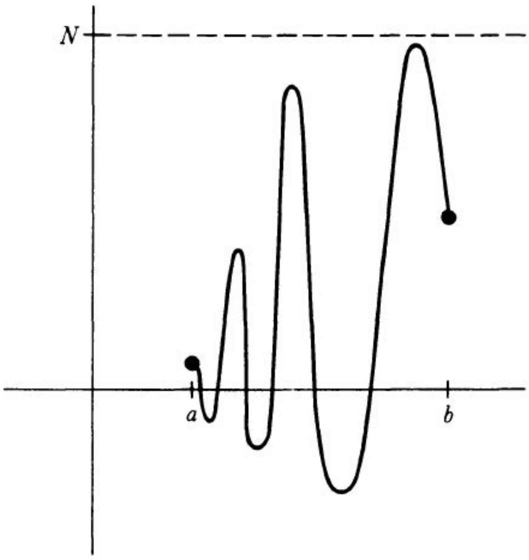


FIGURE 2

THEOREM 3
If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that

$$f(y) >= f(x) \tag{7.4}$$

for all x in $[a, b]$ (Figure 3).

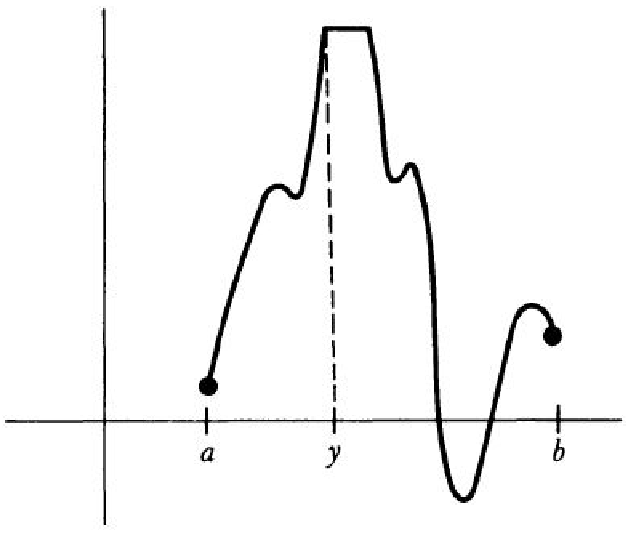


FIGURE 3

5. These three theorems differ markedly from the theorems of Chapter 6.
6. The hypotheses of those theorems always involved continuity at a single point, while the hypotheses of the present theorems require continuity on a whole interval $[a, b]$

7. If continuity fails to hold at a single point, the conclusions may fail.

8. For example, let f be the function shown in Figure 4,

$$f(x) = \begin{cases} -1 & 0 \leq x < \sqrt{2} \\ 1 & \sqrt{2} \leq x \leq 2. \end{cases} \tag{7.5}$$

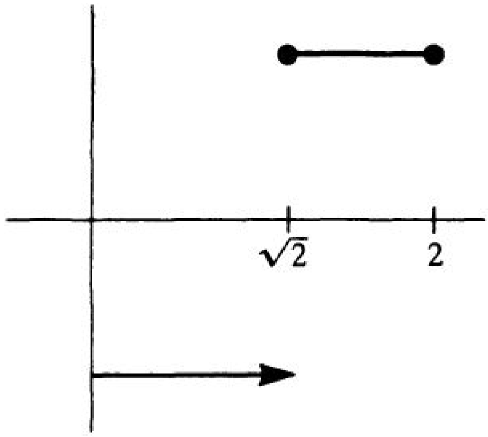


FIGURE 4

9. Then f is continuous at every point of $[0, 2]$ except $\sqrt{2}$, and

$$f(0) < 0 < f(2), \tag{7.6}$$

but there is no point x in $[0, 2]$ such that $f(x) = 0$;

- the discontinuity at the single point $\sqrt{2}$ is sufficient to destroy the conclusion of Theorem 1.

10. Similarly, suppose that f is the function shown in Figure 5,

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases} \tag{7.7}$$

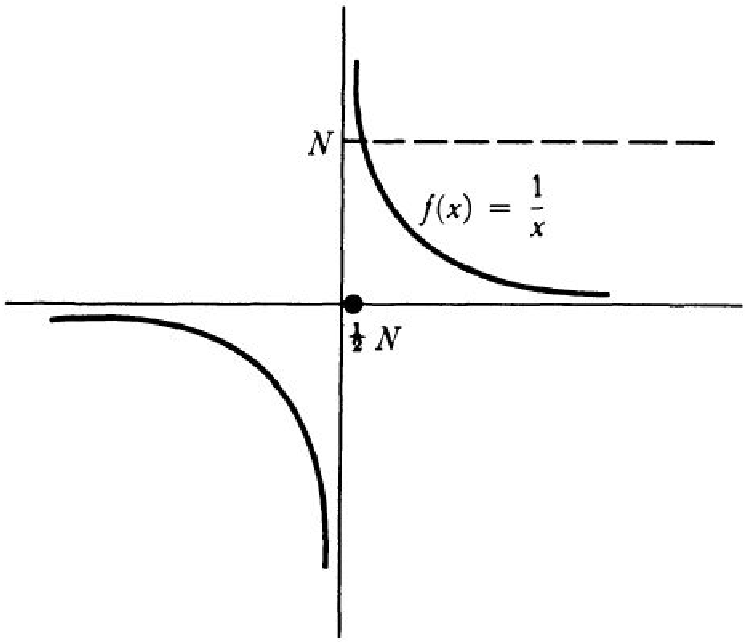


FIGURE 5

11. Then f is continuous at every point of $[0,1]$ except 0, but f is not bounded above on $[0,1]$.
12. In fact, for any number $N > 0$ we have

$$f\left(\frac{1}{2N}\right) = 2N > N. \tag{7.8}$$

13. This example also shows that the closed interval $[a,b]$ in Theorem 2 cannot be replaced by the open interval (a,b), for the function f is continuous on $(0,1)$, but is not bounded there.
14. Finally, consider the function shown in Figure 6,

$$f(x) = \begin{cases} x^2 & x < 1 \\ 0 & x \geq 1. \end{cases} \tag{7.9}$$

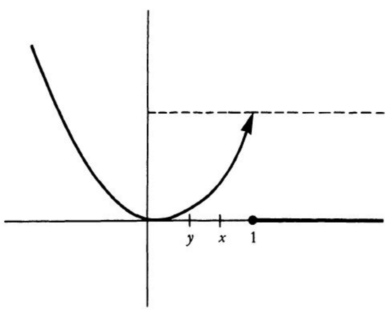


FIGURE 6

THEOREM 1
If f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \tag{7.10}$$

then there is some x in $[a, b]$ such that

$$f(x) = 0. \tag{7.11}$$

THEOREM 2
If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that

$$f(x) <= N \tag{7.12}$$

for all x in $[a, b]$.

THEOREM 3
If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that

$$f(y) >= f(x) \tag{7.13}$$

for all x in $[a, b]$ (Figure 3).

15. On the interval $[0, 1]$ the function f is bounded above, so f does satisfy the conclusion of Theorem 2
16. Even though f is not continuous on $[0, 1]$.
17. But f does not satisfy the conclusion of Theorem 3
    - —there is no y in $[0, 1]$ such that $f(y) >= f(x)$ for all x in $[0, 1]$;
    - In fact, it is certainly not true that
    $$f(1) >= f(x) \tag{7.14}$$
    for all x in $[0, 1]$ so we cannot choose $y = 1$, nor can we choose $0 \leq y < 1$ because $f(y) < f(x)$ if x is any number with $y < x < 1$.
18. This example shows that Theorem 3 is considerably stronger than Theorem 2.
19. Theorem 3 is often paraphrased by saying that a continuous function on a closed interval "takes on its maximum value" on that interval.
20. As a compensation for the stringency of the hypotheses of our three theorems, the conclusions are of a totally different order than those of previous theorems.
21. They describe the behavior of a function, not just near a point, but on a whole interval;
    - such "global" properties of a function are always significantly more difficult to prove than "local" properties, and are correspondingly of much greater power.

22. To illustrate the usefulness of Theorems 1, 2, and 3, we will soon deduce some important consequences, but it will help to first mention some simple generalizations of these theorems.

THEOREM 4
If f is continuous on $[a, b]$ and

$$f(a) < c < f(b), \tag{7.15}$$

then there is some x in $[a, b]$ such that

$$f(x) = c. \tag{7.16}$$

Proof.

1. Let $g = f - c$.
2. Then g is continuous, and g$(a) < 0 < g(b)$.
3. By Theorem 1, there is some x in $[a, b]$ such that $g(x) = 0$. But this means that $f(x) = c$.

□

THEOREM 5
If f is continuous on $[a, b]$ and

$$f(a) > c > f(b), \tag{7.17}$$

then there is some x in $[a, b]$ such that

$$f(x) = c. \tag{7.18}$$

Proof.

1. The function $-f$ is continuous on $[a, b]$ and $-f(a) < -c < -f(b)$.
2. By Theorem 4 there is some x in $[a, b]$ such that $-f(x) = -c$, which means that $f(x) = c$.

□

23. Theorems 4 and 5 together show that f takes on any value between $f(a)$ and $f(b)$. We can do even better than this: if c and d are in $[a, b]$, then f takes on any value between $f(c)$ and $f(d)$.
24. The proof is simple: if, for example, $c < d$, then just apply Theorems 4 and 5 to the interval $[c, d]$.
25. Summarizing, if a continuous function on an interval takes on two values, it takes on every value in between;
    - this slight generalization of Theorem 1 is often called the Intermediate Value Theorem.

THEOREM 6
If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$, that is, there is some number N such that $f(x) \leq N$ for all x in $[a, b]$.

Proof.

1. The function $-f$ is continuous on $[a,b]$, so by Theorem 2 there is a number M such that $-f(x) \leq M$ for all x in $[a,b]$.
2. But this means that $f(x) \leq -M$ for all x in $[a,b]$, so we can let

$$N = -M. \tag{7.19}$$

□

THEOREM 2
If f is continuous on $[a,b]$, then f is bounded above on $[a,b]$, that is, there is some number N such that

$$f(x) <= N \tag{7.20}$$

for all x in $[a,b]$.

26.
- Theorems 2 and 6 together show that a continuous function f on $[a,b]$ is bounded on $[a,b]$, that is, there is a number N such that

$$|f(x)| \leq N \tag{7.21}$$

for all x in $[a,b]$.
- In fact, since Theorem 2 ensures the existence of a number N_1 such that

$$f(x) \leq N_1 \tag{7.22}$$

for all x in $[a,b]$, and Theorem 6 ensures the existence of a number N_2 such that

$$f(x) \leq N_2 \tag{7.23}$$

for all x in $[a,b]$, we can take

$$N = \max(|N_1|, |N_2|) \tag{7.24}$$

THEOREM 7
If f is continuous on $[a,b]$, then there is some y in $[a,b]$ such that $f(y) \leq f(x)$ for all x in $[a,b]$. (A continuous function on a closed interval takes on its minimum value on that interval.)

Proof.

1. The function $-f$ is continuous on $[a,b]$;
   - by Theorem 3 there is some y in $[a,b]$ such that

$$-f(y) \geq -f(x) \tag{7.25}$$

for all x in $[a,b]$, which means that $f(y) \leq f(x)$ for all x in $[a,b]$.

□

THEOREM 8

1. Every positive number has a square root.
2. In other words, if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.

Proof.

1. Consider the function
$$f(x) = x^2, \tag{7.26}$$
which is certainly continuous.
2. Notice that the statement of the theorem can be expressed in terms of f:
   - "the number α has a square root" means that f takes on the value α.
3. The proof of this fact about f will be an easy consequence of Theorem 4.
4. There is obviously a number $b > 0$ such that $f(b) > \alpha$ (as illustrated in Figure 7);

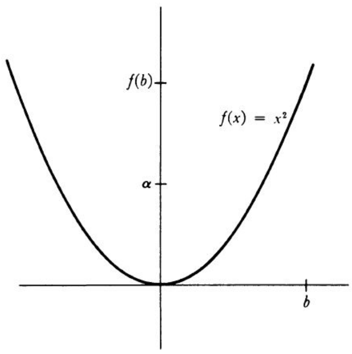


FIGURE 7

5. In fact, if $\alpha > 1$ we can take $b = \alpha$, while if $\alpha < 1$ we can take $b = 1$.
6. Since
$$f(0) < \alpha < f(b), \tag{7.27}$$
Theorem 4 applied to $[0, b]$ implies that for some x (in $[0, b]$), we have
$$f(x) = \alpha, \tag{7.28}$$
i.e.,
$$x^2 = \alpha. \tag{7.29}$$

□

27. Precisely the same argument can be used to prove that a positive number has an nth root, for any natural number n.

28. If n happens to be odd, one can do better: every number has an nth root.

29. To prove this we just note that if the positive number alpha has the nth root x, i.e., if

$$x^n = \alpha, \tag{7.30}$$

then

$$(-x)^n = -\alpha \tag{7.31}$$

(since n is odd), so $-\alpha$ has the nth root $-x$.

30. The assertion, that for odd n any number alpha has an nth root, is equivalent to the statement that the equation

$$x^n - \alpha = 0 \tag{7.32}$$

has a root if n is odd.

31. Expressed in this way the result is susceptible of great generalization.

THEOREM 9
If n is odd, then any equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{7.33}$$

has a root.

Proof.

1. We obviously want to consider the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0; \tag{7.34}$$

we would like to prove that f is sometimes positive and sometimes negative.
2. The intuitive idea is that for large $|x|$, the function is very much like

$$g(x) = x^n \tag{7.35}$$

and, since n is odd, this function is positive for large positive x and negative for large negative x.
3. A little algebra is all we need to make this intuitive idea work.
4. The proper analysis of the function f depends on writing

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right). \tag{7.36}$$

5. Note that

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \tag{7.37}$$

6. Consequently, if we choose x satisfying

$$(*) \quad |x| > 1, 2n|a_{n-1}|, \cdots, 2n|a_0|, \tag{7.38}$$

then

$$|x^k| > |x| \tag{7.39}$$

and

$$\frac{a_{n-k}}{x^k} < \frac{a_{n-k}}{x} < \frac{a_{n-k}}{2n|a_{n-k}|} = \frac{1}{2n}, \tag{7.40}$$

so

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}. \tag{7.41}$$

7. In other words,

$$-\frac{1}{2} \leq \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \leq \frac{1}{2}, \tag{7.42}$$

which implies that

$$\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}. \tag{7.43}$$

8. Therefore, if we choose an $x_1 > 0$ which satisfies (*),then

$$\frac{(x_1)^n}{2} \leq (x_1)^n(1 + \frac{a_{n-1}}{x_1} + \cdots + \frac{a_0}{(x_1)^n}) = f(x_1), \tag{7.44}$$

so that $f(x_1) > 0$.

9. On the other hand, if $x_2 < 0$ satisfies (*), then $(x_2)^n < 0$ and

$$\frac{(x_2)^n}{2} \leq (x_2)^n(1 + \frac{a_{n-1}}{x_2} + \cdots + \frac{a_0}{(x_2)^n}) = f(x_2), \tag{7.45}$$

so that $f(x_2) < 0$.

10. Now applying Theorem 1 to the interval $[x_2, x_1]$ we conclude that there is an x in $[x_2, x_1]$ such that $f(x) = 0$.

□

THEOREM 9
If n is odd, then any equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{7.46}$$

has a root.

32. Theorem 9 disposes of the problem of odd degree equations so happily that it would be frustrating to leave the problem of even degree equations completely undiscussed.

33. At first sight, however, the problem seems insuperable.

34. Some equations, like $x^2 - 1 = 0$, have a solution, and some, like $x^2 + 1 = 0$, do not—what more is there to say?

35. If we are willing to consider a more general question, however, something interesting can be said.

36. Instead of trying to solve the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \tag{7.47}$$

let us ask about the possibility of solving the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c, \tag{7.48}$$

37. for all possible numbers c.

38. This amount to allowing the constant term a_0 to vary.

39. The information which can be given concerning the solution of these equations depends on a fact which is illustrated in Figure 8.

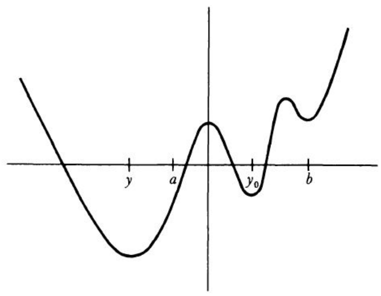


FIGURE 8

40. The graph of the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0, \tag{7.49}$$

with n even, contains, at least the way awe have drawn it, a lowest point.

41. In other words, there is a number y such that

$$f(y) \leq f(x) \tag{7.50}$$

for all numbers x

- — the function f takes on a minimum value, not just on each closed interval, but on the whole line.

42. (Notice that this is false if n is odd.)

43. The proof depends on Theorem 7, but a tricky application will be required.

44.
- We can apply Theorem 7 to any interval $[a, b]$, and obtain a point y_0 such that $f(y_0)$ is the minimum value of f on $[a, b]$;
- but if $[a, b]$ happens to be the interval shown in Figure 8, for example, then the point y_0 will not be the place where f has its minimum value for the whole line.

45. In the next theorem the entire point of the proof is to choose an interval $[a, b]$ in such a way that this cannot happen.

THEOREM 10
If n is even and

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{7.51}$$

then there is a number y such that

$$f(y) \leq f(x) \tag{7.52}$$

for all x.

Proof.

> THEOREM 9
> If n is odd, then any equation
> $$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{7.53}$$
> has a root.

As in the proof of Theorem 9, if

$$M = \max(1, 2n|a_{n-1}|, \cdots, 2n|a_0|), \tag{7.54}$$

then for all x with $|x| \geq M$, we have

$$\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}. \tag{7.55}$$

2. Since n is even, $x^n \geq 0$ for all x, so

$$\frac{x^n}{2} \leq x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = f(x), \tag{7.56}$$

provided that $|x| > M$.

3. Now consider the number $f(0)$.
4. Let $b > 0$ be a number such that

$$b^n > 2f(0) \text{ and also } b > M. \tag{7.57}$$

5. Then, if $x \geq b$, we have (Figure 9)

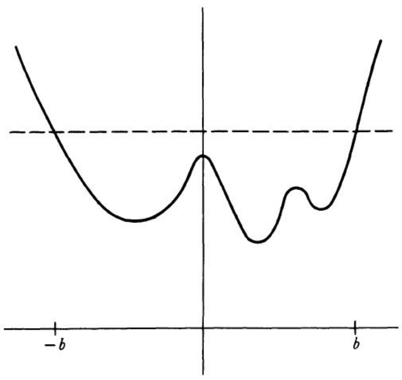


FIGURE 9

$$f(x) \geq \frac{x^n}{2} \geq \frac{b^n}{2} \geq f(0). \tag{7.58}$$

6. Similarly, if $x \leq -b$, then

$$f(x) \geq \frac{x^n}{2} \geq \frac{(-b)^n}{2} = \frac{b^n}{2} \geq f(0). \tag{7.59}$$

7. Summarizing: if
$$x \geq b \text{ or } x \leq -b \tag{7.60}$$
then
$$f(x) \geq f(0). \tag{7.61}$$
8. Now apply Theorem 7 to the function f on the interval $[-b, b]$.
9. We conclude that there is a number y such that (1) if
$$-b \leq x \leq b, \tag{7.62}$$
then
$$f(x) \leq f(x). \tag{7.63}$$
10. In particular,
$$f(y) \leq f(0). \tag{7.64}$$
11. Thus (2) if
$$x \leq -b \text{ or } x \geq b \tag{7.65}$$
then
$$f(x) \geq f(0) \geq f(y). \tag{7.66}$$
12. Combining (1) and (2) we see that
$$f(y) \leq f(x) \tag{7.67}$$
for all x.

□

46. Theorem 10 now allows us to prove thaw following result.

> Theorem 11
>
> 1. consider the equation
> $$(*) \quad x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c, \tag{7.68}$$
> and suppose n is even.
> 2. Then there is a number m such that (*) has a solution for $c \geq m$ and has no solution for $c < m$.

Proof.

1. Let
$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ (Figure 10)}. \tag{7.69}$$

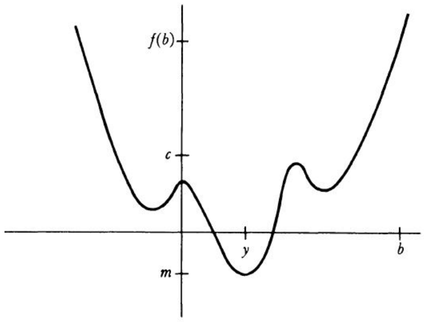


FIGURE 10

2. According to Theorem 10 there is a number y such that

$$f(y) \leq f(x) \tag{7.70}$$

for all x.

3. Let $m = f(y)$.
4. If $c < m$, then the equation (*) obviously has no solution, since the left since the side always has a value $\geq m$.
5. If $c = m$, then (*) has y as a solution.
6. Finally, suppose $c > m$.
7. Let b be a number such that $b > y$ and $f(b) > c$.
8. Then

$$f(y) = m < c < f(b). \tag{7.71}$$

9. Consequently, by Theorem 4, there is some number x in $[y, b]$ such that $f(x) = c$, so x is a solution of (*).

□

47. These consequences of Theorems 1, 2, and 3 are the only ones we will derive now
    - (these theorems will play a fundamental role in everything we do later, however).
48. Only one task remains–to prove Theorems 1, 2, and 3.
49. Unfortunately, we cannot hope to do this
    - — on the basis of our present knowledge about the real numbers (namely, P1-P12) a proof is impossible.

THEOREM 8

1. Every positive number has a square root.
2. In other words, if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.

50. There are several ways of convincing ourselves that this gloomy conclusion is actually the case.

51. For example, the proof of Theorem 8 relies only on the proof of Theorem 1;

52. If we could prove Theorem 1, then the proof of Theorem 8 would be complete, and we would have a proof that every positive number has s square root.

53. As pointed out in Part I, it is impossible to prove this on the basis of P1-P12. Again, suppose we consider the function

$$f(x) = \frac{1}{x^2 - 2}. \tag{7.72}$$

54. If there were no number x with

$$x^2 - 2, \tag{7.73}$$

then f would be continuous, since the denominator would never $= 0$. But f is not bounded on $[0, 2]$.

THEOREM 1
If f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \tag{7.74}$$

then there is some x in $[a, b]$ such that

$$f(x) = 0. \tag{7.75}$$

THEOREM 2
If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that

$$f(x) <= N \tag{7.76}$$

for all x in $[a, b]$.

THEOREM 3
If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that

$$f(y) >= f(x) \tag{7.77}$$

for all x in $[a, b]$ (Figure 3).

55. So Theorem 2 depends essentially on the existence of numbers other than rational numbers, and therefore on some property of the real numbers other than P1-P12.

56. Despite our inability to prove Theorems 1, 2, and 3, they are certainly results which we want to be true,

57. If the pictures we have been drawing have any connection with the mathematics we are doing, if our notion of continuous function corresponds to any degree with our intuitive notion,

58. Theorems 1, 2, and 3 have got to be true.

59. Since a proof of any of these theorems must require some new property of $\mathbb{R}$ which has so far been overlooked, our present difficulties suggest a way to discover that property:

    - let us try to construct a proof of Theorem 1, for example, and see what goes wrong.

60. One idea which seems promising is to locate the first point where $f(x) = 0$, that is, the smallest x in [a,b] such that $f(x) = 0$.

61. To find this point, first consider the set A which contains all numbers x in $[a, b]$ such that f is negative on $[a, x]$.

62. In Figure 11, x is such a point, while x' is not.

63. The set A itself is indicated by a heavy line.

64. Since f is negative at a, and positive at b, the set A contains some points greater than a, while all points sufficiently close to b are not in A.

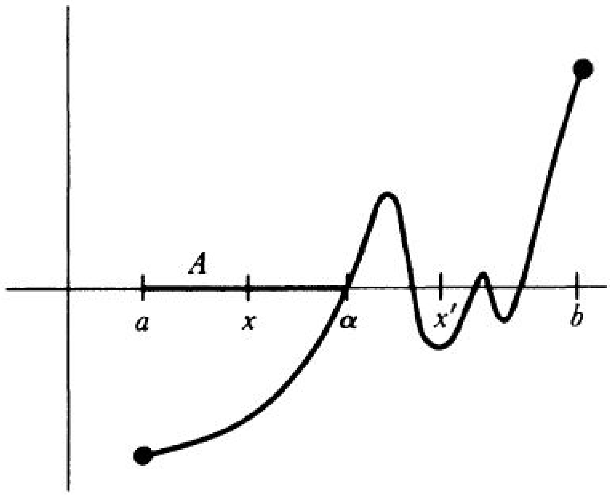


FIGURE 11

65. (We are here using the continuity of f on $[a, b]$, as well as Problem 6-15.)

66. Now suppose alpha is the smallest number which is greater than all members of A; clearly

$$a < \alpha < b. \tag{7.78}$$

67. We claim that $f(\alpha) = 0$, and to prove this we only have to eliminate the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

68. Suppose first that $f(\alpha) < 0$.

69. Then, by Theorem 6-2, $f(x)$ would be less than 0 for all x in a small interval containing α,

    - in particular for some numbers bigger than α (Figure 12);
    - but this contradicts the fact that α is bigger than every member of A,

- since the larger numbers would also be in A.

70. Consequently, $f(\alpha) < 0$ is false.

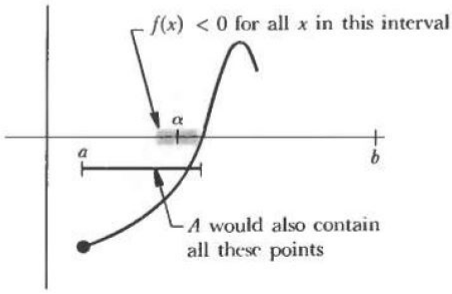


FIGURE 12

71. On the other hand, suppose $f(\alpha) > 0$.
72. Again applying Theorem 6-2, we see that $f(x)$ would be positive for all x in a small interval containing α,
    - in particular for some numbers smaller than alpha (Figure 13).

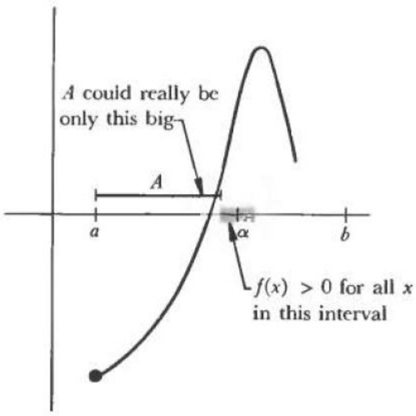


FIGURE 13

73. This means that these smaller numbers are all not in A.
74. Consequently, one could have chosen an even smaller α which would be greater than all members of A.
75. Once again we have a contradiction; $f(\alpha) > 0$ is also false.
76. Hence $f(\alpha) = 0$ and, we are tempted to say, Q.E.D.
77. We know, however, that something must be wrong, since no new properties of $\mathbb{R}$ were ever used, and it does not require much scrutiny to find the dubious point.
78. It is clear that we can choose a number alpha which is greater than all members of A

- (for example, we can choose $\alpha = b$), but it is not so clear that we can choose a smallest one.

79. In fact, suppose A consists of all numbers $x \geq 0$ such that $x^2 < 2$.

80. If the number $\sqrt{2}$ did not exist, there would not be a least number greater than all the members of A;

    - for any $y > \sqrt{2}$ we chose, we could always choose a still smaller one.

7.2 Problem

1. - For each of the following functions, decide which are bounded above or below on the indicated interval, and which take on their maximum or minimum value.
   - (Notice that f might have these properties even if f is not continuous, and even if the interval is not a closed interval.)

   (a) $f(x) = x^2$ on $(-1, 1)$.

   (b) $f(x) = x^3$ on $(-1, 1)$.

   (c) $f(x) = x^2$ on $\mathbb{R}$.

   (d) $f(x) = x^2$ on $[0, \infty)$.

   (e) $f(x) = \begin{cases} x^2 & x \leq a \\ a+2 & x > a, \end{cases}$ on $(-a-1, a+1)$.

   - (It will be necessary to consider several possibilities for a.)

   (f) $f(x) = \begin{cases} x^2 & x \leq a \\ a+2 & x > a, \end{cases}$ on $[-a-1, a+1]$.

   (g) $f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

   (h) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

   (i) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

   (j) $f(x) = \begin{cases} 1, & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

   (k) $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ on $[0, a]$.

   (l) $f(x) = \sin^2\left(\cos x + \sqrt{a + a^2}\right)$ on $[0, a^3]$.

   (m) $f(x) = [x]$ on $[0, a]$.

2. For each of the following polynomial functions f, find an integer n such that $f(x) = 0$ for some x between n and $n+1$.

(a) $f(x) = x^3 - x + 3$.
(b) $f(x) = x^5 + 5x^4 + 2x + 1$.
(c) $f(x) = x^5 + x + 1$.
(d) $f(x) = 4x^2 - 4x + 1$.

3. Prove that there is some number x such that
   (a) $x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} = 119$.
   (b) $\sin x = x - 1$.

4. This problem is a continuation of Problem 3-7.
   (a) If $n - k$ is even, and ≥ 0, find a polynomial function of degree n with exactly k roots.
   (b)
   - A root a of the polynomial function f is said to have multiplicity m if
   $$f(x) = (x - a)^m g(x), \tag{7.79}$$
   where g is a polynomial function that does not have a as a root.
   - Let f be a polynomial function of degree n.
   - Suppose that f has k roots, counting multiplicities, i.e., suppose that k is the sum of the multiplicities of all the roots.
   - Show that $n - k$ is even.

5.
   - Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational.
   - What can be said about f ?

6.
   - Suppose that f is a continuous function on $[-1, 1]$ such that
   $$x^2 + (f(x))^2 = 1 \tag{7.80}$$
   for all x.
   - (This means that $(x, f(x))$ always lies on the unit circle.)
   - Show that either
   $$f(x) = \sqrt{1 - x^2} \tag{7.81}$$
   for all x, or else
   $$f(x) = -\sqrt{1 - x^2} \tag{7.82}$$
   for all x.

7. How many continuous function f are there which satisfy
$$(f(x))^2 = x^2 \tag{7.83}$$
for all x?

8 Least Upper Bound (p.131)

8.1 Context

1. This chapter reveals the most important property of the real numbers.
2. Nevertheless, it is merely a sequel to Chapter 7;
   - the path which must be followed has already been indicated, and further discussion would be useless delay.

Definition

- A set A of real numbers is **bounded above** if there is a number x such that

$$x \geq a \text{ for every } a \text{ in } A. \tag{8.1}$$

- Such a number x is called an **upper bound** for A.

3. Obviously A is bounded above if and only if
   - there is a number x which is an upper bound for A
   - (and in this case there will be lots of upper bounds for A);
   - we often say, as a concession to idiomatic English, that

     "A has an upper bound"

   - when we mean that there is a number which is an upper bound for A.
4. Notice that the term "bounded above" has now been used in two ways
   - —first, in Chapter 7, in reference to functions, and now in reference to sets.
5. This dual usage should cause no confusion, since it will always be clear whether we are talking about a set of numbers or a function.
6. Moreover, the two definitions are closely connected:
7. if A is the set

$$\{f(x) : a < x < b\}, \tag{8.2}$$

   then the function f is bounded above on $[a, b]$ if and only if the set A is bounded above.
8. The entire collection $\mathbb{R}$ of real numbers, and the natural numbers $\mathbb{N}$, are both examples of sets which are not bounded above.
9. An example of a set which is bounded above is

$$A = \{x : 0 < x < 1\} \tag{8.3}$$

10. To show that A is bounded above we need only name some upper bound for A, which is easy enough;
    - for example, 138 is an upper bound for A, and so are 2, $1\frac{1}{2}$, $1\frac{1}{4}$, and 1.

11. Clearly, 1 is the least upper bound of A;
    - although the phrase just introduced is self-explanatory,
    - in order to avoid any possible confusion
    - (in particular, to ensure that we all know what the superlative of "less" means),
    - we define this explicitly.

A number x is a least upper bound of A if

1. x is an upper bound of A
2. if y is an upper bound of A, then $x \leq y$.

12. The use of the indefinite article "a" in this definition was merely a concession to temporary ignorance.
13. Now that we have made a precise definition, it is easily seen that if x and y are both least upper bounds of A, then $x = y$.
14. Indeed, in this case
    - $x \leq y$, since y is an upper bound, and x is a least upper bound.
    - $y \leq x$, since x is am upper bound, and y is a least upper bound.
15. It follows that $x = y$.
16. For this reason we speak of **the** the least upper bound of A.
17. The term **supermum** of A is synonymous and has one advantage.
18. It abbreviates quite nicely to

    sup A (pronounced "soup A").

19. and saves us from the abbreviation

    lub A

    (which is nevertheless used by some authors).
20. There is a series of important definitions, analogous to those just given, which can now be treated more briefly.
21. A set A of real numbers is **boundedbelow** if there is a number x such that

$$x \leq a \text{ for every } a \text{ in } A. \tag{8.4}$$

22. Such a number x is called a lower bound for A.
23. A number x is the greatest lower bound of A if
    - x is a lower bound of A

- if y is a lower bound of A, then $x \geq y$.

24. The greatest lower bound of A is also called the **infimum** of A, abbreviated

$$\text{inf A;}$$

some authors use the abbreviation

$$\text{glb A.}$$

25. One detail has been omitted from our discussion so far
    - — the question of which sets have at least one, and hence exactly one, least upper bound or greatest lower bound.
26. We will consider only least upper bounds, since the question for greatest lower bounds can then be answered easily (Problem 2).
27. If A is not bounded above, then A has no upper bound at all, so A certainly cannot be expected to have a least upper bound.
28. It is tempting to say that A does have a least upper bound if it has *some* upper bound, but, like the principle of mathematical induction, this assertion can fail to be true in a rather special way.
29. If $A = \emptyset$, then A is bounded above.
30. Indeed, any number x is an upper bound for $\emptyset$:

$$x \geq y \text{ for every } y \text{ in } \emptyset. \tag{8.5}$$

31. simply because there is no y in $\emptyset$.
32. Since every number is an upper bound for $\emptyset$, there is surely no least upper bound for $\emptyset$.
33. With this trivial exception however, our assertion is true—and very important, definitely important enough to warrant consideration of details.
34. We are finally ready to state the last property of the real numbers which we need.

(Prop 13)
(The least upper bound property)
If A is a set of real numbers,

$$A \neq \emptyset, \tag{8.6}$$

and A is bounded above, then A has a least upper bound.

35. Property P13 may strike you as anticlimactic, but that is actually one of its virtues.
36. To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it.
37. Of course, the least upper bound property is not really so innocent as all that;

- after all, it does not hold for the rational numbers Q.

38. For example, if A is the set of all rational numbers x satisfying

$$x^2 < 2, \tag{8.7}$$

then there is no rational number y which is an upper bound for A and which is less than or equal to every other rational number which is an upper bond for A.

39. It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.

Theorem 7-1 (IVT)
If f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \tag{8.8}$$

then there is some number x in $[a, b]$ such that

$$f(x) = 0. \tag{8.9}$$

Proof.

1. Our proof is merely a rigorous version of the outline developed at the end of Chapter 7
   - —we will locate the smallest number x in $[a, b]$ with

$$f(x) = 0. \tag{8.10}$$

2. Define the set A, shown in Figure 1, as follows:

$$A = \{x : a < x < b, \text{ and } f \text{ is negative on the interval } [a, x]\} \tag{8.11}$$

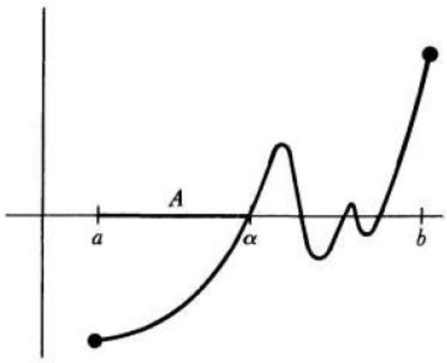


FIGURE 1

3. Clearly $A \neq \emptyset$, since a is in A;
   - in fact, there is some $\delta > 0$ such that A contains all points x satisfying

$$a < x < a + \delta; \tag{8.12}$$

   - this follows from Problem 6-15, since f is continuous on $[a, b]$ and $f(a) < 0$.

4. • Similarly, b is an upper bound for A and,
   • in fact, there is a $\delta > 0$ such that all points x satisfying

$$b - \delta < x < b \tag{8.13}$$

   are upper bounds for A;
   • this also follows from Problem 6-15, since $f(b) > 0$.
5. From these remarks it follows that A has a least upper bound α and that

$$a < \alpha < b. \tag{8.14}$$

6. We now wish to show that $f(a) = 0$, by eliminating the possibilities

$$f(a) < 0 \text{ and } f(a) > 0. \tag{8.15}$$

7. Suppose first that $f(\alpha) < 0$.
8. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for

$$\alpha - \delta < x < \alpha + \delta \tag{8.16}$$

   (Figure 2).

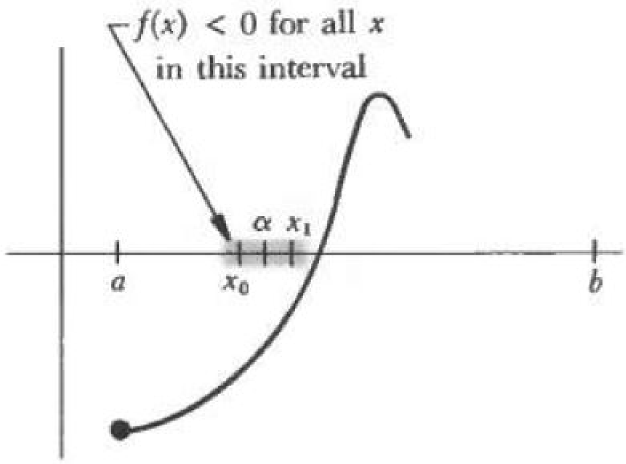


FIGURE 2

9. Now there is some number x_0 in A which satisfies

$$\alpha - \delta < x_0 < \alpha \tag{8.17}$$

   • (because otherwise α would not be the least upper bound of A).
10. This means that f is negative on the whole interval $[a, x_0]$.
11. But if x_1 is a number between α and $\alpha + \delta$, then f is also negative on the whole interval $[x_0, x_1]$.
12. Therefore f is negative on the interval $[a, x_1]$, so x is in A.
13. But this contradicts the fact that α is an upper bound for A;
   • our original assumption that $f(\alpha) < 0$ must be false.

14. Suppose, on the other hand, that $f(\alpha) > 0$.
15. Then there is a number $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 3).

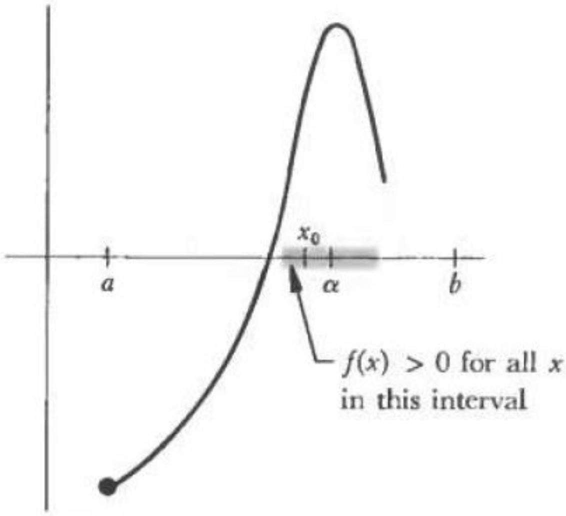


FIGURE 3

16. Once again we know that there is an x_0 in A satisfying

$$\alpha - \delta < x_0 < \alpha; \tag{8.18}$$

but this means that f is negative on $[a, x_0]$, which is impossible, since $f(x_0) > 0$

17. Thus the assumption $f(\alpha) > 0$ also leads to a contradiction, leaving $f(\alpha) = 0$ as the only possible alternative.

□

40. The proofs of Theorems 2 and 3 of Chapter 7 require a simple preliminary result, which will play much the same role as Theorem 6-3 played in the previous proof.

Theorem 1
If f is continuous at a, then there is a number $\delta > 0$ such that f is bounded above on the interval

$$(a - \delta, a + \delta) \tag{8.19}$$

(see Figure 4).

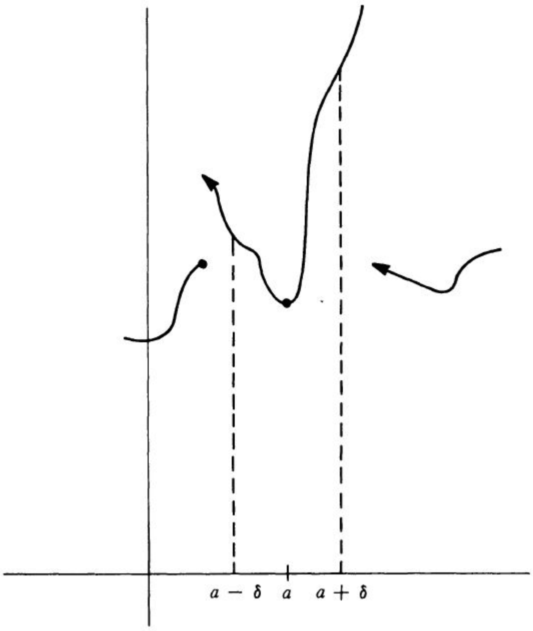


FIGURE 4

Proof.

1. Since $\lim_{x \to a} f(x) = f(a)$, there is, for every $\varepsilon > 0$, a $\delta > 0$ such that, for all x, if

$$|x - a| < \delta, \tag{8.20}$$

then

$$|f(x) - f(a)| < \varepsilon. \tag{8.21}$$

2. It is only necessary to apply this statement to some particular ε (any one will do), for example,

$$\varepsilon = 1. \tag{8.22}$$

3. We conclude that there is a $\delta > 0$ such that, for all x, if

$$|x - a| < \delta, \tag{8.23}$$

then

$$|f(x) - f(a)| < l \tag{8.24}$$

4. It follows, in particular, that if

$$|x - a| < \delta, \tag{8.25}$$

then

$$|f(x) - f(a)| < \varepsilon. \tag{8.26}$$

5. This completes the proof:
   - on the interval

$$(a - \delta, a + \delta) \tag{8.27}$$

the function f is bounded above by $f(a) + 1$.

□

41. It should hardly be necessary to add that we can now also prove that f is bounded below on some interval

$$(a-\delta, a+\delta), \tag{8.28}$$

and, finally, that f is bounded on some open interval containing a.

42. A more significant point is the observation that if

$$\lim_{x \to a^+} f(x) = f(a), \tag{8.29}$$

then there is a $\delta > 0$ such that f is bounded on the set

$$\{x : a \leq x < a+\delta\}, \tag{8.30}$$

and a similar observation holds if

$$\lim_{x \to b^-} f(x) = f(b). \tag{8.31}$$

43. Having made these observations (and assuming that you will supply the proofs), we tackle our second major theorem.

> Theorem 7-2
> If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$.

Proof.

1. Let

$$A = \{x : a \leq x \leq b \text{ and } f \text{ is bounded above on } [a, x]\} \tag{8.32}$$

2. Clearly $A \neq \emptyset$ (since a is in A), and A is bounded above (by b), so A has a least upper bound α.
3. Notice that we are here applying the term "bounded above" both to the set A,
   - which can be visualized as lying on the horizontal axis, and to f,
   - i.e., to the sets

     $$\{f(y) : a \leq y \leq x\}, \tag{8.33}$$

     which can be visualized as lying on the vertical axis (Figure 5).

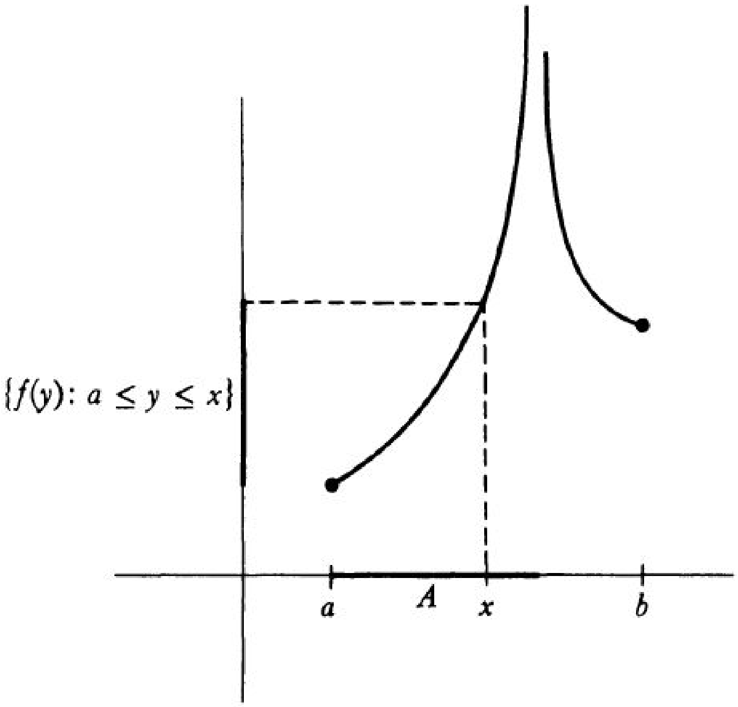


FIGURE 5

4. Our first step is to prove that we actually have

$$a = b. \tag{8.34}$$

5. Suppose, instead, that

$$\alpha < b. \tag{8.35}$$

6. By Theorem 1 there is $\delta > 0$ such that f is bounded on

$$(\alpha - \delta, \alpha + \delta). \tag{8.36}$$

7. Since α is the least upper bound of A there is some x_0 in A satisfying

$$\alpha - \delta < x_0 < \alpha. \tag{8.37}$$

8. This means that f is bounded on $[a, x_0]$.
9. But if x_1 is any number with

$$\alpha - \delta < x_1 < \alpha, \tag{8.38}$$

   then f is also bounded on $[x_0, x_1]$.
10. Therefore f is bounded on $[a, x_1]$, so x_1 is in A, contradicting the fact that α is an upper bound for A.
11. This contradiction shows that

$$a = b. \tag{8.39}$$

12. One detail should be mentioned:

- this demonstration implicitly assumed that $a < \alpha$ [so that f would be defined on some interval

$$(\alpha - \delta, \alpha + \delta); \tag{8.40}$$

the possibility a $= \alpha$ can be ruled out similarly, using the existence of a $\delta > 0$ such that f is bounded on

$$\{x : a \leq x < a + \delta\}. \tag{8.41}$$

13. The proof is not quite complete
    - —we only know that f is bounded on $[a, x]$ for every $x < b$, not necessarily that f is bounded on $[a, b]$.
14. However, only one small argument needs to be added.
15. There is a $\delta > 0$ such that f is bounded on

$$\{x : b - \delta < x \leq b\}. \tag{8.42}$$

16. There is x_0 in A such that

$$b - \delta < x_0 < b. \tag{8.43}$$

17. Thus f is bounded on $[a, x_0]$ and also on $[x_0, b]$, so f is bounded on $[a, b]$.

□

44. To prove that third important theorem we resort a trick.

> Theorem 7-3
> If f is continuity on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$.

Proof.

1. We already know that f is bounded on $[a, b]$, which means that the set

$$\{f(x) : x \text{ in } [a, b]\} \tag{8.44}$$

is bounded.

2. This set is obviously not $\emptyset$, so it has a least upper bound α.
3. Since

$$\alpha \geq f(x) \tag{8.45}$$

for x in $[a, b]$ it suffices to show that

$$a = f(y) \tag{8.46}$$

for some y in $[a, b]$.

4. Suppose instead that

$$\alpha \neq f(y) \tag{8.47}$$

for all y in $[a, b]$.

5. Then the function g defined by

$$g(x) = \frac{1}{\alpha - f(x)},\ x \text{ in } [a,b] \tag{8.48}$$

is continuous on $[a,b]$, since the denominator of the right side is never 0.

6. On the other hand, α is the least upper bound of

$$\{f(x) : x \text{ in } [a,b]\}; \tag{8.49}$$

this means that for every $\varepsilon > 0$ there is x in $[a,b]$ with

$$\alpha - f(x) < \varepsilon \tag{8.50}$$

7. This, in turn, means that for every $\varepsilon > 0$ there is x in $[a,b]$ with

$$g(x) > \frac{1}{\varepsilon}. \tag{8.51}$$

8. But *this* means that g is not bounded on $[a,b]$, contradiction the previous theorem.

□

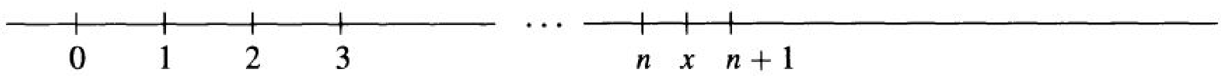
45. At the beginning of this chapter the set of natural numbers $\mathbb{N}$ was given as an example of an unbounded set.

46. We are now going to prove that $\mathbb{N}$ is unbounded.

47. After the difficult theorems proved in this chapter you may be startled to find such an "obvious" theorem winding up our proceedings.

48. If so, you are, perhaps, allowing the geometrical picture of $\mathbb{R}$ to influence you too strongly.

49. "Look, "you may say," the real numbers



Figure

so every number x is between two integers $n, n+1$ (unless x is itself an integer).

50. Basing the argument on a geometric picture is not a proof, however, and even the geometric picture contains an assumption:

    - that if you place unit segments end-to-end you will eventually get a segment larger than any given segment.

51. This axiom, often omitted from a first introduction to geometry, is usually attributed (not quite justly) to Archimedes, and the corresponding property for numbers, that $\mathbb{N}$ is not bounded, is called the *Archimedean property* of the real numbers.

52. This property is not a consequence of P1-P12 (see reference [17] of the Suggested Reading), although it does hold for $\mathbb{Q}$, of course.

53. Once we have P13 however, there are no longer any problems.

> Theorem 2
> $\mathbb{N}$ is not bounded above.

Proof.

1. Suppose $\mathbb{N}$ were bounded above.
2. Since
$$\mathbb{N} \neq \emptyset, \tag{8.52}$$
there would be a least upper bound α for $\mathbb{N}$.
3. Then
$$\alpha \geq n \text{ for all } n \text{ in } \mathbb{N} \tag{8.53}$$
4. Consequently,
$$\alpha \geq n \text{ for all } n \text{ in } \mathbb{N} \tag{8.54}$$
since $n+1$ is in $\mathbb{N}$ if n is in $\mathbb{N}$.
5. But this means that
$$\alpha - 1 \geq n \text{ for all } n \text{ in } \mathbb{N} \tag{8.55}$$
and this means that $\alpha - 1$ is also an upper bound for $\mathbb{N}$, contradicting the fact that α is the least upper bound.

□

54. There is a consequence of Theorem 2 (actually an equivalent formulation) which we have very often assumed implicitly.

> For every $\varepsilon > 0$ there is a natural number n with $\frac{1}{n} < \varepsilon$.

Proof.

1. Suppose not; then
$$\frac{1}{n} \geq \varepsilon \tag{8.56}$$
for all n in $\mathbb{N}$.
2. Thus
$$n \leq \frac{1}{\varepsilon} \tag{8.57}$$
for all n in $\mathbb{N}$.
3. But this means that $\frac{1}{\varepsilon}$ is an upper bound for $\mathbb{N}$, contradicting Theorem 2.

□

55. A brief glance through Chapter 6 will show you that the result of Theorem 3 was used in the discussion of many examples.

56. Of course, Theorem 3 was not available at the time, but the examples were so important that in order to give them some cheating was tolerated.

57. As partial justification for this dishonesty we can claim that this result was never used in the proof of a theorem, but if your faith has been shaken, a review of all the proofs given so far is in order.

58. Fortunately, such deception will not be necessary again.

59. We have now stated every property of the real numbers that we will ever need.

60. Henceforth, no more lies.

8.2 Problem

1. - Find the least upper bound and the greatest lower bound (if they exist) of the following sets.
   - Also decide which sets have greatest and least elements (i.e., decide when the least upper bound and greatest lower bound and greatest lower bound happens to belong to the set)

   (a) $\left\{\frac{1}{n} : n \text{ in } \mathbb{N}\right\}$.

   (b) $\left\{\frac{1}{n} : n \text{ in } \mathbb{Z} \text{ and } n \neq 0.\right\}$.

   (c) $\{x : x = 0 \text{ or } x = \frac{1}{n} \text{ for some } n \text{ in } \mathbb{N}\}$.

   (d) $\{x : 0 \leq x \leq \sqrt{2} \text{ and } x \text{ is rational}\}$.

   (e) $\{x : x^2 + x + 1 \geq 0\}$.

   (f) $\{x : x^2 + x - 1 < 0\}$.

   (g) $\{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$.

   (h) $\left\{\frac{1}{n} + (-1)^n : n \text{ in } \mathbb{N}\right\}$.

2. (a)
   - Suppose $A \neq \emptyset$ is bounded below.
   - Let $-A$ denote the set of all $-x$ for x in A.
   - Prove that $-A \neq \emptyset$ is bounded above, and that $-\sup(-A)$ is the greatest lower bound of A.

   (b)
   - If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A.
   - Show that $B \neq \emptyset$, that B is bounded above, and that $\sup B$ is the greatest lower bound of A.

3. Let f be a continuous function on [a,b] with $f(a) < 0 < f(b)$.

   (a)
   - The proof of the Theorem 1 showed that there is a smallest x in $[a, b]$ with $f(x) = 0$.
   - Is there necessarily a second smallest x in $[a, b]$ with$f(x) = 0$?

- Shoe that there is a largest x in $[a, b]$ with $f(x) = 0$.
- (Try to give an easy proof by considering a new function closely related to f.)

(b)
- The proof of Theorem 1 depend upon consideration of

$$A = \{x : a \leq x \leq b \text{ and } f \text{ is negative on } [a, b]\}. \tag{8.58}$$

- Give another proof of Theorem 1, which depends on consideration of

$$A = \{x : a \leq x \leq b \text{ and } f(x) < 0\}. \tag{8.59}$$

- Which point x in $[a, b]$ with $f(x) = 0$ will this proof locate?
- Give an example where the sets A and B are not the same.

4. Suppose that f is continuous on $[a, b]$ and that $f(a) = f(b) = 0$.

(a)
- Suppose also that $f(x_0) > 0$ for some x_0 in $[a, b]$.
- Prove that there are numbers c and d with a

$$a \leq c < x_0 < d \leq b \tag{8.60}$$

such that

$$f(0) = f(d) = 0, \tag{8.61}$$

but $f(x) > 0$ for all x in (c, d).
- Hint: The previous problem can be used to good advantage.

(b)
- Suppose that f is continuous on $[a, b]$ and that

$$f(a) < f(b). \tag{8.62}$$

- Prove that there are numbers c and d with

$$a \leq c < d \leq b \tag{8.63}$$

such that $f(c) = f(a)$ and $f(d) = f(b)$ and

$$f(a) < f(x) < f(d) \tag{8.64}$$

for all x in (c, d).

5. (a)
- Suppose that $y - x > 1$.
- Prove that there is an integer k such that $x < k < y$.
- Hint: Let l by the largest integer satisfying $l < x$, and consider $l + 1$.

(b)
- Suppose $x < y$.
- Prove that there is a rational number r such that $x < r < y$.
- Hint: If

$$\frac{1}{n} < y - x, \tag{8.65}$$

then

$$ny - nx > 1. \tag{8.66}$$

- (Query: Why have parts (a) and (b) been postponed until this problem set?)

(c)
- Suppose that $r < s$ are rational numbers.
- Prove that there is an irrational number between r and s.
- Hint: As a start, you know that there is an irrational number between 0 and 1.

(d)
- Suppose that $x < y$.
- Prove that there is an irrational number between x and y.
- Hint: It is unnecessary to do any more work; this follows from (b) and (c).

Appendices

A Uniform Continuity

1. Now that we've come to the end of the "foundations," it might be appropriate to slip in one further fundamental concept.
2. This notion is not used crucially in the rest of the book, but it can help clarify many points later on.
3. We know that the function
$$f(x) = x^2 \tag{A.1}$$
is continuous at a for all a.
4. In other words, if a is any number, then for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if
$$|x - a| < \delta, \tag{A.2}$$
then
$$|x^2 - a^2| < \varepsilon. \tag{A.3}$$
5. Of course, δ depends on ε.
6. But δ also depends on a
   - — the δ that works at a might not work at b (Figure 1).
7. Indeed, it's clear that given ε there is no one $\delta > 0$ that works for all a, or even for all positive a.
8. In fact, the number $a + \frac{\delta}{2}$ will certainly satisfy
$$|x - a| < \delta, \tag{A.4}$$
but if $a > 0$, then
$$\left|(a + \frac{\delta}{2})^2 - a^2\right| = \left|a\delta + \frac{\delta^2}{4}\right| \geq a\delta \tag{A.5}$$
and this won't be $< \varepsilon$ once $a > \frac{\varepsilon}{\delta}$.
9. (This is just an admittedly confusing computational way of saying that f is growing faster and faster!
10. On the other hand, for any $\varepsilon > 0$ there will be one $\delta > 0$ that works for all a in any interval $[-N, N)$.
11. In fact, the δ which works at N or $-N$ will also work everywhere else in the interval.
12. As a final example, consider the function
$$f(x) = \sin\left(\frac{1}{x}\right), \tag{A.6}$$
or the function whose graph appears in Figure 18 on page 62.

13. It is easy to see that, so long as $\varepsilon < 1$, there will not be one $\delta > 0$ that works for these functions at all points a in the open interval $(0, 1)$.

14. These examples illustrate important distinctions between the behavior of various continuous functions on certain intervals, and there is a special term to signal this distinction.

DEFINITION
The function f is uniformly continuous on an interval A if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x and y in A, if

$$|x - y| < \delta, \tag{A.7}$$

then

$$|f(x) - f(y)| < \varepsilon. \tag{A.8}$$

15. We've seen that a function can be continuous on the whole line, or on an open interval, without being uniformly continuous there.

16. On the other hand, the function

$$f(x) = x^2 \tag{A.9}$$

did turn out to be uniformly continuous on any closed interval.

17. This shouldn't be too surprising it's the same sort of thing that occurs when we ask whether a function is bounded on an interval

    - — and we would be led to suspect that any continuous function on a closed interval is also uniformly continuous on that interval.

18. In order to prove this, we'll need to deal first with one subtle point.

19. Suppose that we have two intervals $[a, b]$ and $[b, c)$ with the common endpoint b, and a function f that is continuous on (a, c).

20. Let $\varepsilon > 0$ and suppose that the following two statements hold:

    I. if x and y are in $[a, b]$ and $|x - y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$,
    II. if x and y are in $[b, c]$ and $|x–y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$.

21. We'd like to know if there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \tag{A.10}$$

whenever x and y are points in $[a, c]$ with $|x - y| < \delta$.

22. Our first inclination might be to choose δ as the minimum of δ_1 and δ_2.

23. But it is easy to see what goes wrong (Figure 2):

    - we might have x in $[a, b]$ and y in $[b, c)$, and then neither (I) nor (II) tells us anything about

$$|f(x) - f(y)|. \tag{A.11}$$

24. So we have to be a little more cagey, and also use continuity of f at b.

LEMMA

1. Let $a < b < c$ and let f be continuous on the interval $[a, c]$.
2. Let $\varepsilon > 0$, and suppose that statement (I) and (II) hold.
3. Then there is a $\delta > 0$ such that, if
$$|x - b| < \delta_3 \tag{A.12}$$
then
$$|f(x) - f(y)| < \varepsilon. \tag{A.13}$$

Proof.

1. Since f is continuous at b, there is a $\delta_3 > 0$ such that, if
$$|x - b| < \delta_3, \tag{A.14}$$
then
$$|f(x) - f(y)| < \frac{\varepsilon}{2}. \tag{A.15}$$
2. It follows that (III) if
$$|x - b| < \delta_3 \text{ and } |y - b| < \delta_3 \tag{A.16}$$
then
$$|f(x) - f(y)| < \varepsilon. \tag{A.17}$$
3. Choose δ to be the minimum of δ_1 , δ_2 , and δ_3.
4. We claim that this δ works.
5. In fact, suppose that x and y are both in $[b, c]$, then
$$|f(x) - f(y) < \varepsilon| \tag{A.18}$$
by (II).
6. The only other possibility is that
$$x < b < y \quad y < b < x. \tag{A.19}$$
7. In either case, since
$$|x - y| < \delta, \tag{A.20}$$
we also have
$$|x - b| < \delta \text{ and } |y - b| < \delta. \tag{A.21}$$
8. So
$$|f(x) - f(y)| < \varepsilon \tag{A.22}$$
by(III)

□

THEOREM 1
If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof.

1. It's the usual trick, but we've got to be a little bit careful about the mechanism of the proof.
2. For $\varepsilon > 0$ let's say that f is $\varepsilon - good$ on $[a, b]$ if there is some $\delta > 0$ such that, for all y and z in $[a, b]$, if
$$|y - z| < \delta, \tag{A.23}$$
then
$$|f(y) - f(z)| < \varepsilon. \tag{A.24}$$
3. Then we're trying to prove that f is $\varepsilon - good$ on $[a, b]$ for all $\varepsilon > 0$.
4. Consider any particular $\varepsilon > 0$.
5. Let
$$A = x : a \leq x \leq b \text{ and } f \text{ is } \varepsilon - good \text{ on } [a, x]. \tag{A.25}$$
6. Then $A \neq \emptyset$ (since a is in A), and A is bounded above (by b), so A has a least upper bound α.
7. We really should write α_ε, since A and α might depend on ε.
8. But we won't since we intend to prove that $\alpha = b$, no matte what ε is.
9. Suppose that we had $\alpha < b$.
10. Since f is continuous at α, there is some $\delta_0 > 0$ such that, if
$$|y - \alpha| < \delta_0, \tag{A.26}$$
then
$$|f(y) - f(\alpha)| < \frac{\varepsilon}{2}. \tag{A.27}$$
11. Consequently, if
$$|y - \alpha| < \delta_0 \text{ and } |z - \alpha| < \delta_0, \tag{A.28}$$
then
$$|f(y) - f(z)| < \varepsilon. \tag{A.29}$$
12. So f is surely $\varepsilon - good$ on the interval
$$[\alpha - \delta_0, \alpha + \delta_0]. \tag{A.30}$$
13. On the other hand, since α is the least upper bound of A, it $\varepsilon - good$ on
$$[a, a + \delta_0], \tag{A.31}$$
so $a + \delta_0$ is in A, contradiction the fact that α is an upper bound.
14. To complete the proof we just have to show that $\alpha = b$ is actually in A.
15. The argument for this is practically the same:

16. Since f is continuous at b, there is some $\delta_0 > 0$ such that, if

$$|b - y| < \delta_0, \tag{A.32}$$

then

$$|f(y) - f(b)| < \frac{\varepsilon}{2}. \tag{A.33}$$

17. So f is $\varepsilon - good$ on

$$[b - \delta_0, b]. \tag{A.34}$$

18. But f is also $\varepsilon - good$ on

$$[a, b - \delta_0], \tag{A.35}$$

so the Lemma implies that f is $\varepsilon - good$ on $[a, b]$.

□

Advanced Calculus: Fall 2018

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Office Hour: Tuesday 2pm-2:50pm

- References: Books of Zhang, Spivak, Cinlar, and Rudin, and the problem book of Fang.
- Grading: HW 30%, Quizzes 30%, Exams 40%.
- TA: Kuo Li-Feng 0926020285, d0955466322@gmail.com

Calculus starts with a new look at zero!

---

1. If a real number is such that its absolute value is smaller than any positive number, then it must be zero.

---

2. What is the Dirichlet function, and what is the area under its graph?

附件 C

3. What is an upper bound? And lower bound?

---

4. The Least Upper Bound and its existence

Setting:

It is defined by two conditions:

old1.

2.

Existence:

---

5. What are the three hard theorems according to the yellow book?

附件 C

What is a sequence?

-
-

What is a bounded sequence?

---

1. Important Definition: Infinitesimal sequence.

---

2. In words:

---

3. In symbols:

---

4. Geometric meaning:

Write "a sequence is not infinitesimal" in four ways:

1. (Def)

---

2. (In words)

---

3. (In symbols)

---

4. (Geometric meaning)

Example $x_n = \frac{1+(-1)^n}{n}$ is an infinitesimal sequence.

Proof.

1. For any $\epsilon > 0$, we want to choose ??? such that

2. To achieve this, it is **sufficient** to ask ...

   In particular, we choose
$$N = \quad .$$

3. Now the important step: To verify that "The above choice of N will do the job!"

Example let $a \in \mathbb{R}$ s.t. $a > 1$. Then $s_n = \frac{1}{a^n}$ is infinitesimal.

Proof.

1. For any $\epsilon > 0$, we want to choose ??? such that

2. To achieve this, it is **sufficient** to ask ...

   In particular, we choose
$$N = \quad .$$

3. Now the important step: To verify that "The above choice of N will do the job!"

Three Important Questions Before Calculus

1. What is very small?
2. What is very large?
   - Or, what is infinity?
   - Or, how to define a number called “infinity”?
3. What is “very close to a number”?

Now, please write a mathematical definition for each of the above!

1.

2.

3.

Now give your own examples for the above three!

附件 C

HW: (Due on Sept. 18) (Recall that you need to do at least four problems from below.)

1. Define "inf". Moreover,
   - What is its other name?
   - Then show that it is unique if existing.
   - Then give an example such that it does not exist.

   (Note that you need to give a complete definition. A complete definition includes set-up!)

2. - Write "a sequence is unbounded" in terms of symbols!
   - Write "a sequence is not infinitesimal" in terms of symbols!

3. Show that
   - $c_n = n + (-1)^n n$ is unbounded.
   - $a_n = \frac{n+(-1)^n}{n^2}$ is infinitesimal.

4. Show that $a_n = \frac{n-2018}{\frac{1}{10^7}n^5 - n + 1}$ is infinitesimal.

5. Theorem (p. 51):

   (A) If both $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal, then so is $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$.

   (B) If $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal and $\{\beta_n\}_{n=1}^{\infty}$ is bounded, then $\{\alpha_n\beta_n\}_{n=1}^{\infty}$ is infinitesimal.

After having done at least four problems above, choose any problems from Chapter 2 of the problem book.

附件 C

(107-09-18) Important definition

1. Definition: The limit of a sequence (p. 55) ♣

$$\lim_{n\to\infty} x_n = a,$$

that is, we say that the limit of a sequence is a real number a if

- For any ....
- there exists ... such that
- for any n ...

Equivalent Defintion: In terms of $x_n - a$:

---

2. In words:

---

3. In symbol:

---

4. Geometric interpretation

Theorem 1 (p. 57) The limit of a sequence, if existing, is unique.

附件 C

Example 3 (p. 59) Let $a > 1$. Show that $\sqrt[n]{a} = 1$.

Example 4 (p. 60) Show that $\sqrt[n]{n} = 1$.

附件 C

Basic Properties of Convergent Sequences

Theorem 4 (p. 61) A convergent sequence is bounded.

Before the proof, we introduce the set-up:

Theorem 5 (pp. 61-62) (1) If $\lim x_n = a$, then $\lim |x_n| = |a|$.

---

(2) If $\lim x_n = a, \lim y_n = b$, then $\lim x_n \pm y_n = a \pm b$.

---

(3) If $\lim x_n = a, \lim y_n = b$, then $\lim x_n y_n = ab$.

附件 C

(4) If $\lim x_n = a$, then $\lim \frac{1}{x_n} = \frac{1}{a}$. (But what conditions we are missing?)

Please write a complete statement for the above result:

Proof.

Now please do the same for $f(x) = \sqrt{x}$.

附件 C

Theorem 6. (p. 68) If $\lim x_n < \lim y_n$, then there is an $N \in \mathbb{N}$ s.t. for all $n > N$, we have

$$x_n < y_n.$$

Show that the converse is not true!

Theorem 7. (p. 69) If both x_n and y_n are convergent, and there is an N_0 s.t.

$$x_n \leq y_n, \quad \forall n > N_0,$$

then $\lim x_n \leq \lim y_n$.

Proof. By contraction.

附件 C

HW: (Due on Sept. 25) (Recall that you need to do at least four problems from below.)

1. If $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, then so is

$$\beta_n = \frac{\alpha_1 + \cdots + \alpha_n}{n}, \quad n = 1, 2, \cdots .$$

2. If $\{\alpha_n \geq 0\}_{n=1}^{\infty}$ is infinitesimal, then so is

$$\beta_n = \sqrt[n]{\alpha_1 \cdots \alpha_n}, \quad n = 1, 2, \cdots .$$

3. The sequence $z_n = \sqrt[n]{\frac{1}{n!}}$ is infinitesimal.

4. Write a complete statement for "if $\lim x_n = a$, then $\sqrt{x_n} = \sqrt{a}$"; then prove it.

5. Do the same for $f(x) = \sqrt[3]{x}$.

6. Do the same for $f(x) = \sin x$.

7. Let $a \in (0, 1)$. Show that $\sqrt[n]{a} = 1$ without using the case $a > 1$.

After having done at least four problems above, choose any problems from Chapter 2 of the problem book.

附件 C

(107-09-25, Tuesday) Chapter 2, Section 3: Convergence Principles

Increasing Sequence:

Decreasing Sequence:

Strictly increasing Sequence:

Strictly decreasing Sequence:

Important assumption on sup **and** inf**:**

Theorem 1. The necessary and sufficient condition for an increasing sequence to be convergent is that it is bounded above.

In symbols:

Corollary (the easy-to-remember version): A bounded, monotone sequence is ???

Proof of Theorem 1. We need to treat "necessity" and "sufficiency" separately.

♣ **Necessity:**

♣ **Sufficiency:**

附件 C

Example 1. (p. 72) Let $a > 0$. Find the limit

$$\lim \frac{a^n}{n!}.$$

IMPORTANT:

- We do not know whether the limit exists.
- To find the limit includes to find out whether it exists. This is usually more difficult than find the numerical answer.
- But for now, this is usually easy since we only have one good tool. (Q: What is it?)

Proof. We have three steps. (Often you may want to switch the first two steps.)

1. The sequence is bounded (above or below?).

---

2. the sequence is monotone.

---

3. Now we know that the limit exists. So we can assume

$$\lim x_n = a.$$

Next is the easy & fun step:

附件 C

Example 2. (p. 73) Let $x_1 = \sqrt{2}$, $x_2 = \sqrt{2+\sqrt{2}}, \cdots$, $x_n = \sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}, \cdots$. Find $\lim x_n$.

Example: Show that $\lim \sin(n)$ DNE. (Q: What is DNE?) (Then do the same for ???)

附件 C

Theorem 2 (Nested Invervals, p. 76) Let

$$I_n = [a_n, b_n] \supset I_{n+1} = [a_{n+1}, b_{n+1}], \qquad n \geq 1$$

be a sequence of nested closed intervals in $\mathbb{R}$.

If

$$\lim(b_n - a_n) = 0,$$

then

$$\cap_{n\geq 1} I_n \neq \phi.$$

In particular,

- We necessarily have the intersection to be a singleton. Let it be c.
- We necessarily have
$$\lim a_n = c = \lim b_n.$$
- c is the only real number satisfying the following:
$$a_n \leq c \leq b_n, \quad \forall n \in \mathbb{N}.$$

Proof.

1. Claim A: $\{a_n\}$ is increasing and bounded above.

2. Similarly, $\{b_n\}$ is decreasing and bounded below.
3. By ... we have ...

4. Moreover, these two limits are the same because ...

---

5. Also recall that the above is limit can also be written as

6. Now we can conclude that the intersection is nonempty because

---

7. The intersection is a singleton because

附件 C

Theorem 4 (Bolzano-Weierstrass, p. 79) A bounded equence has a convergence subsequence.

In symbols:

Proof.

1. We regard $[a, b]$ as the 0^{th} generation and we bisect it to obtain the next generation.

2. We get the even next generation, that is, the 2^{nd} generation similarly:

3. In general, we obtain the k^{th} generation by

4. Summary so far: We have a sequence of nested intervals:

5. Now we apply ...... to get

6. Claim: $\{x_n\}$ has a subsequence convergent to c.

7. We pick x_{n_1} by

8. We pick x_{n_k} by

9. **Claim:**

$$\lim_{k\to\infty} x_{n_k} = c. \qquad (\clubsuit)$$

10. This is simple:

11. In more details via $\epsilon - N$:

附件 C

HW: (Due on Oct. 2) (Recall that you need to do at least four problems from below.)

1. Show that $\sqrt[n]{n} = 1$.

2. Let $a > 0, x_0 > 0$. The sequence $\{x_n\}$ is defined iteratively:

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right), \quad n = 1, 2, \cdots .$$

   Please find $\lim x_n$. (Hint: Two parts!)

3. Show that the limit of following sequence exists:

$$x_n = \left(1 + \frac{1}{n}\right)^n, \ n = 1, 2, \cdots .$$

4. State and prove the nested interval theorem.

5. State and prove the Bolzano-Weierstrass theorem.

6. Show that $\lim \tan(n)$ DNE.

After having done at least four problems above, choose any problems from pp. 15-18 of Chapter 2 of the problem book.

附件 C

Warm-up exercises:

1. How to describe a subsequence in symbols?

---

2. Show that if a sequence is convergent, then any of its subsequence converges to the same limit. (Please set up the notations carefully.)

   Set-up:

   Proof.

♣ **Cauchy sequence (important):**

Non-Cauchy in symbols:

Fact: A Cauchy sequence is bounded.

附件 C

♣ **Cauchy's convergence principle:**

$$\text{Convergence} \;=\; \text{Cauchy}.$$

A complete statement:

The advantage of being Cauchy:

Proof: It has two directions. Which is easier?

附件 C

Determine whether the following sequences are Cauchy:

1. $x_n = (-1)^n$

It is clearly non-Cauchy, which means, in symbols:

---

2. $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$

附件 C

3. $x_n = \frac{1}{n^2}$

---

4. $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$

---

5. $x_n = \sin(n)$

Definition: We say that $\lim_{n\to\infty} x_n = +\infty$ if

In this case we also say that $\{x_n\}$ is divergent to $+\infty$, or simply ∞.

Converse:

Definition: We say that $\lim_{n\to\infty} x_n = -\infty$ if

Theorem: An increasing sequence is either congvergent or divergent to ∞.

State in symbols: An non-increasing sequence.

Exercise: Show that if $x_n \to \infty$, then $x_n^n \to \infty$.

附件 C

Arithmetics of infinities: (Q: What does this mean?)

-
-
-
-

We analyze $\frac{\infty}{\infty}$. How?

This is how: ∞ is not a number; instead, it is represented by a sequence.

That is, we assume that $\lim x_n = \lim y_n = \infty$, and we consider

$$\lim \frac{x_n}{y_n} = \text{Everything can happen!}$$

For "everything", how many situations can you think of ?

-
-
-
-

Seven indefinite forms: (also called indeterminate forms)

- There is no need to do addition.
- Subtraction:
- Multiplication:
- Division-1:
- Division-2:
- Power-1:
- Power-2:
- Power-3:

附件 C

Exercise: If $f(x) \leq g(x)$, $\forall x \in D$, then

$$\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x).$$

Exercise: If $f(x) \leq g(x)$, $\forall x \in D$, then

$$\sup_{x\in D} f(x) + g(x) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$

Exercise: Let $f(x)$ be defined on D. Then

$$\sup_{x\in D}\{-f(x)\} = -\inf_{x\in D} f(x).$$

HW: (Due on Oct. 9) (Recall that you need to do at least four problems from below.)

- Please latex one of your homework problems, and it will be counted as two problems this week. (You need to print out both your latex code and the pdf file.)
- Please remember to use A4 paper, and switch to a new page for a new problem. (We have add enough sub-problems so you cannot write two problems in one page.

1. Use the Cauchy criteron to determine convergence/divergence for
   - $\sum_{k=1}^{\infty} \frac{1}{k}$
   - $\sum_{k=1}^{\infty} \frac{1}{k^2}$
2. - Let $x_n > 0$ for each $n \geq 1$. Prove that $\lim x_n = \infty$ iff $\lim \frac{1}{x_n} = 0$.
   - Show that $x_n \to 0$ iff $|x_n| \to 0$.
   - Show that $x_n \to 0$ iff $x_n^2 \to 0$.
3. Explain the following in symbols:
   (a) What is a non-convergence sequence?
   (b) $\lim_{n\to\infty} x_n = \infty$.
   (c) $\lim_{n\to\infty} x_n = -\infty$.
   (d) x_n does not diverge to ∞. (Hint: This includes two cases:
   - x_n has a finite limit, or
   - x_n has no finite limit.

   But you are suggested not to give your solution case by case, but to give just one condition. You can start with $\lim_{n\to\infty} x_n = \infty$ and find its opposite.)
4. Use two methods to show that if a sequence is not bounded, then it has a subsequence convergent to ∞ or $-\infty$. (Hint: One direct proof and one by contradiction.)
5. (a) What are the seven indefinite forms?
   (b) Then use your own examples to show that each of them can be nontrivial. (When grading, we will ask you to modify your examples on the spot.)

After having done at least four problems above, choose any problems from pp. 15-24 of Chapter 2 of the problem book.

附件 C

(Important) Limits of a function:

$$\lim_{x \to a} f(x) = A$$

If $A = f(a)$**, i.e.,**

$$\lim_{x \to a} f(x) = f(a),$$

then we say that $f(x)$ is continuous at a.

We divide the implication of "continuity" into two parts:

-
-

There are two ways to define the limit of a function: Heine & Cauchy.

♣ Heine's sequential definition:

♣ Cauchy's $\epsilon - \delta$ definition: If $x_0 \in \mathbb{R}, \eta > 0$, then let

$$U(x_0, \eta) =$$

$$\check{U}(x_0, \eta) =$$

Now the definition:

What is a neighborhood of ∞**?**

Then, $-\infty$**?**

Practice: Define $\lim_{x \to \infty} f(x) = A$. (In general there are nine possibilities.)

附件 C

A: Use Cauchy's definition to show $\lim_{x\to a}\sin(x) = \sin(a)$.

B: Use Cauchy's definition to show $\lim_{x\to 4}\sqrt{x} = 2$.

Two basic lemmas:

♣ **Lemma (p. 104)** Let $a, A \in \mathbb{R}$. If

$$\lim_{x \to a} f(x) = A,$$

then there is an $\eta > 0$ such that f is bounded on $\check{U}(a, \eta)$.

Proof.

♣ ♣ **Lemma (p. 104)** Let $a, A(\neq 0) \in \mathbb{R}$. If

$$\lim_{x \to a} f(x) = A,$$

then there is an $\eta > 0$ such that $|f|$ is bounded below on $\check{U}(a, \eta)$. Indeed, we can have for any $x \in \check{U}(a, \eta)$,

$$|f(x)| > \frac{|A|}{2}.$$

Proof.

附件 C

Now we prove Theorem 5 (p. 103):

$$\text{Heine} = \text{Cauchy}$$

♣ **Heine ⇒ Cauchy:** First set-up, which I consider to be an important part of the problem.

Given:

Want:

By contradiction (in warm-up), we want:

For $\forall n \in \mathbb{N}$, we choose

$$\delta = \delta_n = \frac{1}{n}.$$

Then by our contradiction assumption, we have

♣ ♣ **Heine ⇐ Cauchy:**

Given:

Want:

Our goal is

$$f(x_n) \to A.$$

So for any $\epsilon > 0$, we need to find an N s.t. $\cdots$

附件 C

Set-up: A continuous function on a closed interval.

Theorem 1 (Intermediate Value Theorem: special case) Suppose that $f(x)$ is continuous on $[a, b]$. If $f(a)f(b) < 0$, then there exists

$$c \in (a, b) \quad \text{s.t.} \quad f(c) = 0.$$

Proof. WLOG, we assume that $f(a) < 0 < f(b)$.

1. Consider the middle point $\frac{a+b}{2}$.

   If   , then

   If not, then ...

2. Now we choose $[a_1, b_1]$ with the following property:

   -
   -
   -

3. In general, we can choose $[a_k, b_k]$ such that

   -
   -
   -

4. Now we continue the process. There are two possibilities:

   -
   -

---

5. By the nested interval theorem,

6. Because $f(x)$ is continuous,

附件 C

HW: (Due on Oct. 16) (Recall that you need to do at least four problems from below.)

1. Show that a decreasing sequence is either convergent or divergent to $-\infty$.
2.
   - Define uniform convergence;
   - then give an interesting example and verify it;
   - then give an example with escape of mass and non-uniform convegence. (Verify it of course.)
3. State the converse of the following six:

   (1) the convergence of sequence,

   (2) bounded sequence,

   (3) Heine on the limit of a function,

   (4) Cauchy on the limit of a function,

   (5) Heine on the continuity of a function at a point, and

   (6) Cauchy on the continuity of a function at a point.
4. Use at least three methods to show that

$$\sup_{x\in D}\{-f(x)\} = -\inf_{x\in D} f(x),$$

   here $f(x)$ is defined on D.
5. Use $\epsilon - \delta$ to define the nine possibilities of $\lim_{x\to\Delta} f(x) = \#$.
6. Use $\epsilon - \delta$ to show $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.
7. Use $\epsilon - \delta$ to show
   - $\lim_{x\to\frac{\pi}{4}} \tan(x) = 1$
   - $\lim_{x\to -1} \frac{1}{x} = -1$.

After having done at least four problems above, choose any problems from p. 20, or afterwards, of the problem book.

附件 C

In this page we discuss a few consequences of IVT.

Example 1 (Brouwer's Fixed Point Theorem) If $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point. That is, $\exists c \in [a, b]$, such that

$$f(c) = c.$$

Proof. Consider a new function

$$g(x) = \qquad .$$

Example 2 The equation $x^3 - 2x - 5 = 0$ has a root in $(2, 3)$.

Theorem 2 (The Intermediate Value Theorem, or IVT) A continuous function on a closed interval assumes any values between its endpoint values.

♣ **Please translate the above into symbols!**

Proof. So easy!

Fact: Let $n \in \mathbb{N}$ be odd. Then the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

has at least a root. (Hint: Go yellow)

附件 C

Theorem 3 A continuous function on a closed interval is bounded.

♣ **Translate please!**

Proof. By contradiction. So we assume that ????

1. We obtain $I_1 = [a_1, b_1]$ as follows:

2. Continue, and we get

3. By the nested interval theorem, we get

4. Because $f(x)$ is continous at c, apply our easy lemma on p. 104 (which one?),

5. Now we apply the definition of sequential convergence to $\lim a_n = c$ and $\lim b_n = c$.

Example 3. Does $f(x) = 1/x$ violate the above theorem?

Example 4. Does $f(x) = \frac{\sin x}{x}$ violate the above theorem?

附件 C

The max-min theorem (p. 125) A continuous function on a closed interval achieves its sup and inf.

Translate!

Proof.

1. Let $M = \sup_{x\in[a,b]} f(x), \quad m = \inf_{x\in[a,b]} f(x)$. Then
$$m, M \in \mathbb{R}.$$
   Why?

2. **Claim ♠:** We can find a sequence $\{x_n\}_{n=1}^{\infty} \subset [a,b]$ such that
$$M - \frac{1}{n} < f(x_n) \leq M.$$
   Why?

3. Bolzano-Weierstrass says that ...

4. **Claim ♣:** The above x' does the job!

Important little thing: What is the difference between sup and max?

Now please reformulate the above theorem in terms of max and min:

附件 C

Exercise A: Let $f_n(x) = \frac{x^n}{1+x^n}$. Determine whether it is uniformly convergent on $[0, 1]$.

Exercise B: Let $f_n(x) = \frac{x^n}{1+x^n}$. Determine whether it is uniformly convergent on $[0, 1/2]$.

Exercise C: Let $f_n(x) = \frac{x^n}{1+x^n}$. Determine whether it is uniformly convergent on $[2, \infty)$.

附件 C

(107-10-18) Recall the definition of continuous functions

i. Set-up:

ii. $f(x)$ is a continuous function if

iii. This means the key formula:

iv. The above formula indeed contains three statements:

-
-
-

♣ Define the continuity of a function at a point via $\epsilon - \delta$:

♠ Important observation: The δ above may depend on the point x_0.

Q: What does this mean?

Define uniform continuity in two ways:

Set-up: Let $E \subset \mathbb{R}$ be a subset. Let f be defined on E.

1. $\epsilon - \delta$:

---

2. Sequential:

The uniform continuity theorem:

Example 5 (p. 126) Let $E = \mathbb{R}$, $f(x) = x$. Yes or No? Please try both definitions.

Example 6 (p. 127) Let $E = \mathbb{R}$, $f(x) = x^2$. Yes or No? Please try both definitions.

♣ **How about** $E = \mathbb{R}$, $f(x) = \sin(x)$**?**

附件 C

The uniform continuity theorem:

Proof. By contradiction. we assume ???

1. We assume that

2. As before, we choose $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then we get

3. Apply B-W to $\{x'_n\}$:

4. Claim: $x''_{n_k} \to x_0$ also!

5. Apply continuity of $f(x)$ at x_0:

附件 C

Theorem 6 (p. 128) Let $f(x)$ be defined on $E \subset \mathbb{R}$. Then $f(x)$ is uniformly continuous iff for any

$$\lim(x_n - y_n) = 0,$$

where $\{x_n\} \subset E$ and $\{y_n\} \subset E$, we have

$$\lim[f(x_n) - f(y_n)] = 0. \qquad (\clubsuit)$$

Proof. We need to prove both "necessity" and "sufficiency".

1. First, necessity. That is, we assume

   and we consider

   and we want ($\clubsuit$):

2. Apply the definition of

$$\lim(x_n - y_n) = 0,$$

3. Then we recall the definition of our goal: $\lim[f(x_n) - f(y_n)] = 0$.

---

4. Second, sufficiency. That is,
   - We assume the sequential condition ($\clubsuit$) is satisfied already.
   - We want the $\epsilon - \delta$ condition to be satisfied also.

5. By contradiction, we assume

6. But by $\delta_n = \frac{1}{n}$, we know that

$$\lim(x_n - y_n) = 0.$$

   So the sequential condition ($\clubsuit$) ...

附件 C

Example: Show that $f(x) = \frac{1}{x}$ on $(0, 1)$ is not uniformly continuous.

Method 1:

Method 2:

Example: Show that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$ by any method.

Then use one of the two definitions to show that $f(x) = x^2$ is uniformly continuous on $[1, 3]$.

附件 C

HW: (Due on Oct. 23) (Recall that you need to do at least four problems from below.)

1.
   - Define one-sided limits.
   - Prove that the limit of a function exists at a point iff both one-sided limits exist at that point and the two one-sided limits are indeed equal.
2.
   - Let $f_n(x) = \frac{x^2}{n}$ be defined on $[0, 100000000000000000000]$ for each $n \geq 1$. Prove that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent.
   - Let $g_n(x) = \frac{x^2}{n}$ be defined on $[0, \infty)$. Prove that $\{g_n\}_{n=1}^{\infty}$ is not uniformly convergent.
3. Any two exercises between # 2.5.7 and #2.5.14. (So if you do all these eight exercises, it will be counted as four problems.)
4. Is $y = \sqrt[3]{x}$ uniformly continuous on $[0, \infty)$? Justify your answer please.

After having done at least four problems above, choose any problems on uniform continuity in the problem book.

(107-10-23) Next HW (Due on October 30)

- #2.4.1, #2.4.2, #2.4.3, and any exercises between # 2.5.7 and #2.5.14 which you haven't done.

Derivative and differentiability

Derivative: This is so easy! But the set-up first please!

Differentiability:

Here we recall the definition of $o(h)$:

Theorem 3 (p. 162) A function f has a derivative at a point x iff it is differentiable at that point.

Proof. Sufficiency first. That is, we assume ...

and we want:

Necessity now. That is, we assume ...

and we want:

附件 C

Theorem 4 (p. 163) If f is differentiable at x, then it is also continuous at x.

Proof.

Remark: Give an example to show that the converse is not true.

Differentials: Suppose that $f(x)$ is differentiable at x_0. Then we introduce two notations:

- $dx =$
- $dy =$

Two ways to interpret the Leibnitz notation $\frac{dy}{dx}$:

-
-

Write the base rules of differentiability in terms of differentials:

- $\pm$
- $\star$
- $\div$

Exercise: Find $d\big(e^x \sin(2x)\big) =$

附件 C

(107-10-25) Parametric equations

The best example for parametric equations is for the circle:

$$x^2 + y^2 = 1.$$

Exercise: Re-write it in a parametric form.

The general form of a parametric equation:

Exercise: Write the equation of a straight line in parametric form.

But first ask yourself: What is a straight line?

Method 1:

Method 2:

Do the same for the parabola.

Do the same for the hyperbola.

附件 C

Q: What is the composition rule for derivatives?

Q: What is the composition rule for derivatives in terms of differentials?

Derivatives of parametric equations:

Set-up:

Want:

Equation of tangent lines for parametric equations:

Parametric form of the equation of tangent lines for parametric equations:

Example: $x = \cos^3(t),\ y = \sin^3(t)$ for $t = \frac{\pi}{3}$.

附件 C

Polar curve: Find the derivative formula for a polar curve.

Set-up:

Want:

Exercise: Let

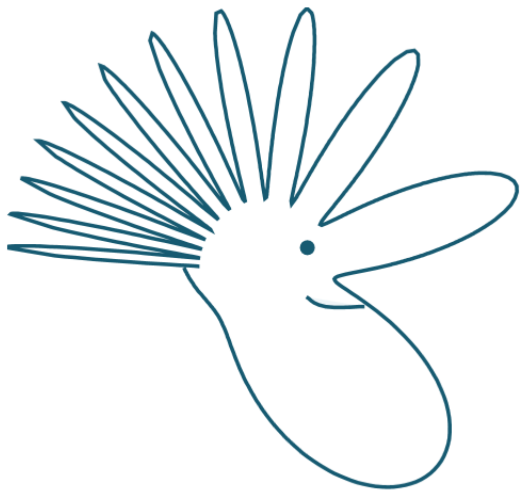
$$x = a\Big(\ln\tan\frac{t}{2} + \cos t\Big), \quad y = a\sin t, \qquad (a > 0, 0 < t < \pi).$$

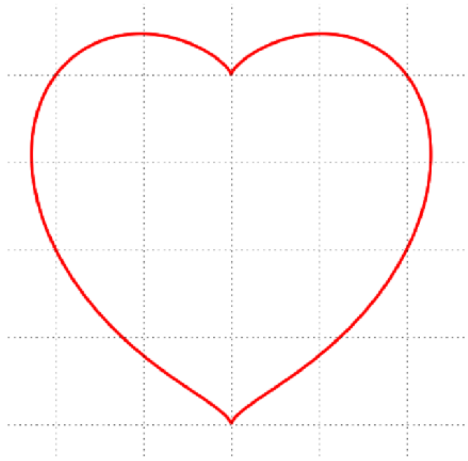
- Let A be any point on the curve,
- let L be the tangent line at A, and
- let B be the intersection between L and the x-axis.

Find the length of the segment $|AB|$.

附件 C

Find the quations for the following curves, and then find an example of tangent line for each of them.





Hint: https://www.intmath.com/plane-analytic-geometry/8-curves-polar-coordinates.php

https://tpenguinltg.wordpress.com/2014/02/15/representing-the-heart-shape-precisely/

Exercise: Analyze the curve to see why $r = \sin(2^\theta) - 1.7$ matches one of the above curve.

Hard work: Try to figure out what the following curves look like!

1. $r = 1 - \sin(\theta)$
2. $(x^2 + y^2 + x)^2 = x^2 + y^2$
3. $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$

Plot them if you know how to use a software!

附件 C

3.a Infinitesimal increment formula (p187)

Differentiability: Recall what does it mean for $f(x)$ to be differentiable at x_0?

Interval:

Interior point:

Local max:

Strict local max:

Extrema:

Extreme points:

Lemma: Let $A \in \mathbb{R}$ and $A \neq 0$. If we have

$$\varphi(h) = Ah + o(h), \quad (h \to 0),$$

then for sufficiently small $h \neq 0$, $\varphi(h)$ and Ah have the same sign.

We explain the fine points carefully:

- $o(h)$:
- sufficiently small $h \neq 0$:
- the same sign:

Proof.

附件 C

Fermat's theorem: If f has an extreme value at x_0, then $f'(x_0) = 0$.

Rigorous statement:

Proof. By contradiction.

Critical points:

Example: Extrema but not critical.

Example: Critical but not extremal.

Theorem 2 (p. 189) The closed interval method.

Example: Use the CIM to find the absolute max and min of $f(x) = \frac{x+1}{x^2+1}$ over $\mathbb{R}$.

Rolle's theorem: Let $f(x)$ be continuous over $[a, b]$ and differentiable over (a, b), and satisfy

$$f(a) = f(b).$$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

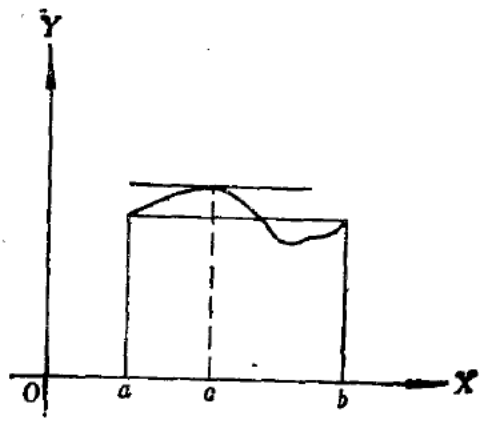


图 4-7

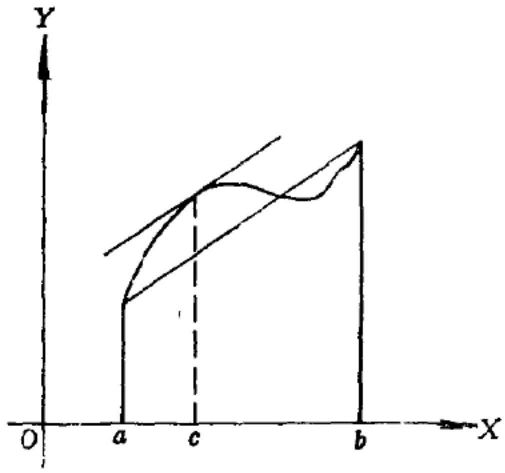


图 4-8

Lagrange's theorem: (Also know as the "mean value theorem", or MVT.)

Applications of MVT:

Theorem 5: $f' = 0$ iff

Corollary:

Example 1: If $f''(x) = 0$ over $\mathbb{R}$, then

Q: Does the above conclusion hold if $\mathbb{R}$ is replaced by an interval?

Example 2: if $f^{(n)}(x) = 0$ over $\mathbb{R}$, then

Theorem 6. If $f(x)$ is continuous on an interval I, and differentiable on I^0, then

"$f(x)$ is increasing on I iff $f'(x) \geq 0$ on I^0."

附件 C

The first sufficient condition for extrema.

Recall $U(x_0, \eta)$ and $\check{U}(x_0, \eta)$:

Statement: If $f(x)$ is increasing but not strictly increasing on an interval, then

---

Suppose $f(x)$ is continuous on a neighborhood $U(x_0, \eta)$ and differentiable on $\check{U}(x_0, \eta)$.

(1) If $f'(x)(x - x_0) > 0$ for all $x \in \check{U}(x_0, \eta)$, then f has a strict ? at x_0.

(2) If $f'(x)(x - x_0) < 0$ for all $x \in \check{U}(x_0, \eta)$, then f has a strict ? at x_0.

Proof.

The second sufficient condition for extrema.

Suppose that $f(x)$ is defined on an interval I, $x_0 \in I^0$ is an interior point, $f''(x_0)$ exists, and

$$f'(x_0) = 0.$$

(1) If $f''(x_0) > 0$, then f has a strict ? at x_0.

(2) If $f''(x_0) > 0$, then f has a strict ? at x_0.

附件 C

HW: (Due on Nov. 13) (Recall that you need to do at least four problems from below.)

1. $f_2(x) = \sqrt{x}\ln(x)$ is uniformly continuous on $(0, \infty)$.
2. $f_3(x) = x\ln(x)$ is not uniformly continuous on $(0, \infty$.
3. Show that $e^x = ax^2 + bx + c$ has at most three roots.
4. Let $f(x)$ be differentiable on $\mathbb{R}$. Show that between two roots of $f(x)$ there is a root for $f(x) + f'(x)$.
5. Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + a_n = 0$. Then prove that the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ has at least one root in $(0, 1)$.
6. Show that the Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

   has n roots in $[-1, 1]$.
7. What is the Hermite polynomial $H_n(x)$? Show that it has n distinct roots.

After having done at least four problems above, choose any problems on or after p. 40 in the problem book.

附件 C

Chapter 8, Section 1: Cauchy's MVT and L'hopital's rule

Theorem (Cauchy's MVT)

- Let $f(x), g(x) \in C[a,b]$, and assume that they are differentiable on (a,b).
- Moreover, we assume that $g'(x) \neq 0$ for all $x \in (a,b)$.
- Then we claim that there exists $\xi \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof 1: Geometry

1. Consider the curve given (f,g). What does this mean?

2. Now use the idea of the standard MVT to find a tangent such that ...

Proof 2: Reduce to Rolle's by an auxillary function

1. In order to use Rolle's/MVT, we construct a new, auxillary function by

2. It is well defined because

3. Now we use Rolle's

附件 C

L'Hopital's rule for $\frac{0}{0}$ (Version 1):

- Let $a, L \in \mathbb{R}$.
- Let $f(x), g(x)$ be defined on some (punctured) neighborhood $U(a, \eta)$ of a, such that

$$f(a) = g(a) = 0.$$

- Assume that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

- Then we claim that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Very easy by Cauchy's MVT.

Example: $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

L'Hopital's rule for $\frac{0}{0}$ (Version 2): Let $a = \infty$ and $L \in \mathbb{R}$.

L'Hopital's rule for $\frac{0}{0}$ (Version 3): Let $L = \infty$ and $a \in \mathbb{R}$.

L'Hopital's rule for $\frac{0}{0}$ (Version 4): Let $a = \infty$ and $L = \infty$.

附件 C

L'Hopital's rule for $\frac{\infty}{\infty}$ (Version 1): Let $a, L \in \mathbb{R}$.

- Let $f(x), g(x)$ be defined on some (punctured) neighborhood $U(a, \eta)$ of a, such that

$$\lim_{x \to a} g(x) = \infty.$$

- Then $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ implies $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Proof. Not very easy, but still by Cauchy's MVT. **KEY:** We want to represent the target $\frac{f(x)}{g(x)}$ in terms of $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}$, so we can use Cauchy's MVT. But how? (Hard or easy?)

$$\frac{f(x)}{g(x)} = \left(1 - \frac{g(x_0)}{g(x)}\right)\frac{f'(\xi)}{g'(\xi)} + \frac{f(x_0)}{g(x)}. \qquad (\clubsuit)$$

Reduction: We claim that it is sufficient to assume $L = 0$. This is indeed easy.

---

1. For any $\epsilon > 0$, choose $\delta_1 > 0$ such that

$$\left|\frac{f'(\xi)}{g'(\xi)}\right| < \frac{\epsilon}{3}, \quad \forall |\xi - a| < \delta_1.$$

2. Fix $x_0 = a + \frac{\delta_1}{2}$.

---

3. Observe that

$$\lim_{x \to a} \frac{g(x_0)}{g(x)} = \lim_{x \to a} \frac{f(x_0)}{g(x)} = 0,$$

so there exists a $\delta_2 > 0$ such that

$$\left|\frac{g(x_0)}{g(x)}\right| < \frac{1}{2} \quad \text{and} \quad \left|\frac{f(x_0)}{g(x)}\right| < \frac{\epsilon}{2}, \quad \forall |x - a| < \delta_2.$$

4. Now we choose

$$\delta = \min\{\frac{\delta_1}{2}, \delta_2\}.$$

We can use $\epsilon - \delta$ to show $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$.

附件 C

Easy but important examples:

1. $\lim_{x\to\infty} \frac{x^k}{e^x}$, here $k \in \mathbb{N}$.

---

2. $\lim_{x\to\infty} \frac{x^\pi}{e^{\sqrt{2}x}}$.

---

3. $\lim_{x\to\infty} \frac{\ln x}{x^a}$, here $a > 0$.

---

4. $\lim_{x\to\infty} \frac{(\ln x)^b}{x^a}$, here $a, b > 0$.

More examples, recommended for memorization.

- $\lim_{x\to 0} \frac{\sin x}{x}$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x}$
- $\lim_{x\to 0} \frac{\sin x - x}{x^3}$
- $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$
- $\lim_{x\to 0} \frac{\ln(1+x) - x}{???}$

附件 C

Streamelined proof of the L'Hopital's rule for $\frac{\infty}{\infty}$ (Version 1): Let $a, L \in \mathbb{R}$.

- Let $f(x), g(x)$ be defined on some (punctured) neighborhood $U(a, \eta)$ of a, such that

$$\lim_{x\to a} g(x) = \infty.$$

- Then $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ implies $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

---

a. **[Proof.]** We claim that it is sufficient to assume $L = 0$.

b. So our goal now is to show that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

c. For any $\epsilon > 0$, choose $\delta_1 > 0$ such that

$$\left|\frac{f'(\xi)}{g'(\xi)}\right| < \epsilon, \quad \forall 0 < |\xi - a| < \delta_1.$$

d. Now we choose an auxiliary point

$$x_0 = a + \frac{\delta_1}{2}.$$

e. Pause and draw a pic please!

f. Observe that

$$\lim_{x\to a} \frac{g(x_0)}{g(x)} = \lim_{x\to a} \frac{f(x_0)}{g(x)} = 0,$$

because ...

so there exists a $\delta_2 > 0$ such that

$$\left|\frac{g(x_0)}{g(x)}\right| < 1 \quad \text{and} \quad \left|\frac{f(x_0)}{g(x)}\right| < \epsilon, \quad \forall 0 < |x - a| < \delta_2.$$

(Better another pic.)

g. Now we choose

$$\delta = \min\{\frac{\delta_1}{2}, \delta_2\}.$$

We can use $\epsilon - \delta$ to show $\lim_{x\to a^+} \frac{f(x)}{g(x)} = 0$. Indeed, we shall show that

$$\left|\frac{f(x)}{g(x)}\right| < 3\epsilon, \qquad \forall\ a < x < a + \delta.$$

h. Now the two complicated formulas show up finally:

附件 C

HW: (Due on Nov. 20) (Recall that you need to do at least four problems from below.)

1. State and prove another version of L'Hopital's rule of $\frac{\infty}{\infty}$ type.
2. State the two definitions of continuity of a function at a point and prove their equivalence.
3. State the two definitions of convergence of a sequence and prove their equivalence.
4. State the two definitions of u.c. and prove their equivalence.
5. Show that the following function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{1}$$

has derivatives of any order on $\mathbb{R}$.

After having done at least four problems above, choose any problems on or after p. 50 in the problem book.

附件 C

Applications of L'Hopital's rule:

A: Recall the infinitesimal increment formula and prove it by the L'Hopital's rule.

B: Suppose that $\varphi(x)$ is defined on $U(a,\eta)$, and

$$\varphi(a) = \varphi'(a) = \cdots = \varphi^{(n)}(a) = 0.$$

Then

$$\varphi(x) = o[(x-a)^n].$$

C: Write an f, g version of the above (B).

D: Important exercise: for any f, find an example of a polynomial for g.

From L'Hopital's to Maclaurin

We start with $\sin(x)$, and we know

$$\sin(x) \sim x \quad \text{as } x \to 0.$$

We express this as a limit:

Now we consider $\sin(x) - x$, and we know by the above

$$\sin(x) - x = o(x) \ (Q : Why?)$$

Now we examine whether we have

$$\sin(x) - x \sim x^2 \quad \text{or} \quad \sin(x) - x \sim x^3.$$

Important observation:

- We should ask the first case for x^2 first.
- We ask the case of x^3 because we already know its answer.

Now write down the corresponding limits and the $o(\cdot)$ expressions!

Time to push it to x^4 and x^5!

Time to push it to infinity! The answer is called 'Maclaurin". Or, the more precise statement is ...

附件 C

Do everything over from the previous page for $\cos(x)$

How to find the Maclaurin series of a function, suppose we know that it exists!

This is by L'Hopital's again!

Assume $f(x) \sim c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$.

Now to find c_0?

Now to find c_1?

Now to find c_2?

Now to find c_n?

附件 C

Examples of Maclaurin series

1. The general formula

$$f(x) = \qquad\qquad\qquad\qquad + o(x^{n+1}).$$

Set-up:

2. $\ln(1 + x)$ (Important)

3. $\ln(1 + 2x)$

4. $\ln(2 + x)$

5. $\ln(2 + x^2)$

附件 C

Q: Why does the Maclaurin series of an odd function have only odd powers?

Rigorous formulation:

Proof.

Q: Why can we expand $f'(x)$ when $f(x)$ is hard, like for

$$f(x) = \arctan(x) \sim \qquad\qquad ???$$

Rigorous formulation:

Proof.

附件 C

HW: (Due on Nov. 27) (Recall that you need to do at least four problems from below.)

1. State and prove any version of L'Hopital's rule for $\frac{\infty}{\infty}$.
2. State and prove the Taylor expansion formula with Peano remainders.
3. State and prove any version of the monotone convergence principle.
4. State and prove the intermediate value theorem.
5. True or false, and why? (Hint: It is hard.)
   - Let $f(x)$ be differentiable at each point in $(0, 1)$, that is, $f'(x)$ exists for each $x \in (0, 1)$.
   - Let $f'(0.1) > 0$ and $f'(0.9) < 0$.
   - Q: Can we always find $c \in (0.1, 0, 9)$ such that
   $$f'(c) = 0?$$
6. The above continued:
   - Is the above problem in contradiction with IVT?
   - Please write a general version of the above problem.

After having done at least four problems above, choose any problems on or after p. 50 in the problem book.

附件 C

Shortcoming of the Peano remainder:

The general remainder: $R_{n+1}(x) =$

The Lagrange remainder:

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \qquad (\clubsuit)$$

We re-write in terms of

$$\theta = \frac{\xi - a}{x - a}.$$

The complete statement:

Lemma: Please invent your lemma here!

Proof.

附件 C

Integral remainder and Cauchy remainder:

1. Start with

$$\psi(1) = \psi(0) + \int_0^1 \psi'(t)dt.$$

Now apply IBP to see what you can get:

---

2. We hope to obtain

$$\psi(1) = \psi(0) + \psi'(0) + \int_0^1 \psi''(t)(1-t)dt.$$

Now we try to obtain a similar formula for ψ'''.

---

3. The general formula is:

---

4. Proof by induction:

---

5 (The integral remainder) Now apply to $f(x)$, a good time to test yourself!

---

6 (The Cauchy remainder)

附件 C

Examples: three remainders

A. $f(x) = e^x$

The general formula for the first remainder:

The general formula for the second remainder:

The general formula for the third remainder:

B. $f(x) = \sin(x)$

The first remainder:

The second remainder:

The third remainder:

附件 C

Taylor series vs. Taylor polynomials

Lemma

1. Let I be an open interval and $f(x) \in C^\infty(I)$ and $a \in I$.
2. Recall that
$$R_{n+1}(x) =$$
3. If there are positive numbers H, Q and $N \in \mathbb{N}$ such that
$$|f^{(n)}(x)| \leq HQ^n, \qquad \forall x \in I,\ n > N,$$
then
$$\lim_{n\to\infty} R_{n+1}(x) = 0, \qquad \forall x \in I.$$
4. That is,
$$f(x) =$$

Proof. We use remainder:

附件 C

A. Find the interval of convergence for the Maclaurin series of $f(x) = e^x$.

B. Find the interval of convergence for the Maclaurin series of $f(x) = \sin(x)$.

C. Find the interval of convergence for the Maclaurin series of $f(x) = \ln(1 + x)$.

HW: (Due on Nov. 27 ???) (Recall that you need to do at least four problems from below.)

1. Show that e is irrational.
2. State and prove the Taylor expansion formula with integral remainders.
3. Explain how to deduce the three remainder formulas for $f(x) = e^x$.
4. Explain how to deduce the three remainder formulas for $f(x) = \cos(x)$.
5. Explain how to deduce the three remainder formulas for $f(x) = \ln(x)$.
6. Find the interval of convergence of the Maclaurin series for $f(x) = \frac{1}{1+x}$.
7. Find the interval of convergence of the Maclaurin series for $f(x) = (1+x)^\pi$.
8. Find the interval of convergence of the Maclaurin series for $f(x) = \tan(x)$.
9. True or false, and why? (Hint: It is hard.)
   - Let $f(x)$ be differentiable at each point in $(0,1)$, that is, $f'(x)$ exists for each $x \in (0,1)$.
   - Let $f'(0.1) > 0$ and $f'(0.9) < 0$.
   - Q: Can we always find $c \in (0.1, 0, 9)$ such that
   $$f'(c) = 0?$$
10. The above continued:
    - Is the above problem in contradiction with IVT?
    - Please write a general version of the above problem.

After having done at least four problems above, choose any problems on or after p. 50 in the problem book.

附件 C

Practice: Find the Maclaurin series up to given order:

1. $\ln\left(\frac{1+x}{1-2x}\right) \qquad (x^n)$

---

2. $\frac{x^2}{\sqrt{1-x+x^2}} \qquad (x^4)$

---

3. $\frac{1}{\sqrt{1-x^2+x^4}} \qquad (x^4)$

---

4. $\ln(1+x+x^2+x^3) \qquad (x^6)$ [Practice: If you get it done quickly, then the same for

$$\ln(1+x+x^2+x^3+x^4) \quad \text{and} \quad \ln(1+x+x^2+x^3+x^4+x^5).$$

Then, is it meaningful to keep going up? How about going down?]

Six Steps to graph a function: Take $f(x) = e^{-x^2}$ as an example.

1. Domain

2. Even, odd, or periodic?

3. Asymptotes

4. Solve $f'(x) = 0$ and determines the signs of $f'(x)$ in between. Draw a table!

x	$\left(-\frac{1}{\sqrt{2}}, 0\right)$	0	$\left(0, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, +\infty\right)$
y'					
y''					
y					
备注					

5. Solve $f''(x) = 0$ and determines the signs of $f''(x)$ in between. Draw in the table!

6. Plot the intersections with x, y-axises, and plot other important points.

附件 C

Example 4 (p. 59) $f(x) = \frac{2x}{1+x^2}$

1. Domain

2. Even, odd, or periodic?

3. Asymptotes

4. Solve $f'(x) = 0$ and determines the signs of $f'(x)$ in between. Draw a table here!

x	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, +\infty)$
y'							
y''							
y							
备注							

5. Solve $f''(x) = 0$ and determines the signs of $f''(x)$ in between. Draw in the table above!

6. Plot the intersections with x, y-axises, and plot other important points.

Example 5 (p. 60) $f(x) = \frac{x^2}{1+x^2}$

附件 C

Example 6 (p. 62) $f(x) = \frac{x^3}{x^2-1}$

1. Domain

2. Even, odd, or periodic?

3. Asymptotes

4. Solve $f'(x) = 0$ and determines the signs of $f'(x)$ in between. Draw a table here!

x	$(-1, 0)$	0	$(0, 1)$	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, +\infty)$
y'						
y''						
y						
备注						

5. Solve $f''(x) = 0$ and determines the signs of $f''(x)$ in between. Draw in the table above!

6. Plot the intersections with x, y-axises, and plot other important points.

附件 C

(107-12-06 Thursday) Riemann sums & Definite integrals

Partition:

Refinement:

Modulus of P:

Sample points ξ:

The Riemann sum $\sigma(f, P, \xi)$ It has three arguments!

Upper and Lower (Darboux) sums $L(f, P)$ and $U(f, P)$:

Key Fact: Suppose that a partition Q refines P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof.

In particular, when we keep refining the partitions,

- L is increasing and
- U is decreasing.

Moreover, if P_1, P_2 are any two partitions, what can we say about

$$P_1 \cup P_2?$$

附件 C

Three Definitions of Riemann Integrals

Goal: To take the limit of Riemann sums.

Infinitely fine: We say that a sequence of partitions $\{P^{(n)}\}$ is infinitely fine if $|P^{(n)}| \to 0$.

Set-up: Given $f(x)$ defined on $[a, b]$.

1. Sequential: We say that a real number $I \in \mathbb{R}$ is the definite integral of $f(x)$ on $[a, b]$ if

---

2. $\epsilon - \delta$: We say that a real number $I \in \mathbb{R}$ is the definite integral of $f(x)$ on $[a, b]$ if

---

3. We recall that L is inceasing and U is decreasing when refining. So we define the upper and lower integrals to be

$$\overline{\int_a^b} f(x)dx =$$

$$\underline{\int_a^b} f(x)dx =$$

If they agree, then ...

附件 C

The most famous example:

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \cap \mathbb{Q}^c \end{cases} \tag{2}$$

Solution I:

Solution II:

Solution III:

附件 C

Example: Show that $f(x) = x^2$ on $[0, 1]$ is Riemann integrable and find its integral.

Property: Suppose that f and g are integrable on $[a, b]$, then prove that

$$f + g$$

is also integrable. Then find its integral. (Hint: Try to use three methods!)

附件 C

Lemma: If $f(x)$ on $[a, b]$ is Riemann integrable, then f is bounded.

Proof by contradiction.

1. We choose $\epsilon = 1$. Then we can find δ such that

2. Given a partition P, if f on $[a, b]$ is unbounded, then

3. Now we choose our $\xi = \{\xi_1, \cdots, \xi_n\}$ by

Fact: If $f(x)$ on $[a, b]$ is integrable, and $g(x)$ differs from $f(x)$ at finitely many points, then $g(x)$ is also integrable.

附件 C

Easy properties/questions:

1. If f on $[a,b]$ is integrable, and $\lambda \in \mathbb{R}$, then

Proof.

---

2. If $f(x)$ is integrable on $[1,5]$, then so is it on $[2,4]$. But is it obvious? **In particular, can we use the additivity of integrals?**

---

3. We define $\int_2^1 f(x)dx$ to be

---

4. Does it make sense to define $\int_1^1 f(x)dx$? What does this say about open integrals like $(0,1)$?

---

5. State and prove the monotonicity of integrals. (So you need to figure out what is the statement!)

---

6. State and prove the squeezing principle for integrals.

---

7. State and prove the mean value theorem for integrals.

附件 C

Theorem 2 Let $a < b < c$. If $f(x)$ is integrable on both $[a, b]$ and $[b, c]$, then $f(x)$ is integral on $[a, c]$. Moreover, ...

Proof.

1. First by the above Lemma, we have

2. If we use the sequential definition, we start with ???

3. Now we choose a new partition $\tilde{P}$ and the associated sample points $\tilde{\xi}$ such that

$$b \in \tilde{P}.$$

4. Now we compare $\sigma(f, P, \xi)$ and $\sigma(f, \tilde{P}, \tilde{\xi})$. ***(This is the heart of the matter!)***

5. The only different term is about

$$f(\xi_k)(x_k - x_{k-1}),$$

   which is replaced by

6. Now the partition $\tilde{P}$ induces two partitions on $[a, b]$ and $[b, c]$, respectively, and we have

$$\sigma(f, \tilde{P}, \tilde{\xi}) = \qquad\qquad\qquad\qquad .$$

   Now we look at the limit on each side!

附件 C

§ 6.2(p236) Newton-Leibniz formula, a.k.a. the "Fundamental Theorem of Calculus."

Theorem: Suppose that $f \in C[a,b]$. If there exists another function $F(x)$ such that

- $F \in C[a,b]$, and
- F is differentiable on (a,b), and
- $F'(x) = f(x)$ for all $x \in (a,b)$,

then $f(x)$ is integrable on $[a,b]$, and

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. The proof uses MVT and the uniform continuity theorem. Recall them now please!

---

1. To show the integrability, we need to start with an arbitrary ...

2. We apply MVT to $F(b) - F(a) = \sum_{i=1}^{m} \Big[F(x_i) - F(x_{i-1}) \Big]$

---

3. Now what is the difference between the above, and our target in the Riemann sum?

4. By the uniform continuity theorem, we choose δ to be ...

5. The last step by the $\epsilon - \delta$ definition of integrals:

附件 C

HW: (Due on December 18) (Recall that you need to do at least four problems from below.)

1. Show two of the three definitions of integrals are equivalent.
2. Show another two of the three definitions of integrals are equivalent.
3. Use three methods to show that if f and g are integrable on $[a, b]$, then prove that
$$f + g$$
is also integrable.
4. Use two definitions to show that if $f(x)$ on $[a, b]$ is integrable, and $g(x)$ differs from $f(x)$ at finitely many points, then $g(x)$ is also integrable.
5. Show that if f and g are integrable on $[a, b]$, then prove that $f \cdot g$ is also integrable. (Hint: hard)
6. Explain how to deduce the three remainder formulas for $f(x) = \tan(x)$.
7. Let $a < b < c$. If $f(x)$ is integrable on both $[a, b]$ and $[b, c]$, then show that $f(x)$ is integral on $[a, c]$. Then, what is the "moreover" statement?
8. Show that $f(x) \in C[a, b]$ is Riemann integrable.
9. Show that an increasing function on a closed interval is Riemann integrable.
10. If $f(x)$ on $[a, b]$ is integrable, and $g(x)$ differs from $f(x)$ at finitely many points, then $g(x)$ is also integrable.
11. If $f(x)$ is integrable on $[1, 5]$, then so is it on $[2, 4]$.
12. State and prove FTC.

After having done at least four problems above, choose any problems on or after p. 75 in the problem book.

Practice for integrals:

A: Deduce the formula for the lateral surface area of a conical frustum. You need to set up all notations carefully!

B: Find the surface area of an ellipsoid. You need to set up all notations carefully!

C: Find the arc length of $y = e^x$ over $[0, 2018]$.

D: Find the arc length of $y = x^3$ over $[0, 107]$.

E: Find the volume enclosed by the cylinder $x^2 + y^2 = 1$ and two planes $z = 0$ and $z = 2(x + 1)$.

F: Use two methods to obtain a recursive formula for $I_n = \int_0^1 \frac{x^n}{1+x} dx$.

G: Find the surface area of a sphere between two parallel planes. You need to set up all notations carefully.

H: Use a third method to obtain a recursive formula for $I_n = \int_0^1 \frac{x^n}{1+x} dx$.

I:

(a) Write a formula for area under a function, but assum that the function is given by the parametric form.

(b) Write a general formula for area enclosed by a closed curve given by the parametric form.

(c) Find the area enclosed by $x = 2t - t^2, y = 2t^2 - t^3$.

J: Show that

$$\int_0^{\sqrt{2\pi}} \sin(x^2) dx > 0.$$

K: Let $f \in C[a, b]$ be increasing. Use three methods to show that

$$\int_a^b x f(x) dx \geq \frac{a+b}{2} \int_a^b f(x) dx.$$

L: Suppose that $f(x)$ on $\mathbb{R}$ is bounded and differentiable, and it satisfies

$$|f(x) + f'(x)| \leq 1, \quad x \in \mathbb{R},$$

then show that

$$|f(x)| \leq 1.$$

M: Let $f(x)$ on $[0, \infty)$ be differentiable, and satisfy

$$0 \leq f'(x) \leq f(x), \; f(0) = 0.$$

Show that $f(x) = 0$.

N: Show that

$$\lim_{n \to \infty} \int_n^{n+2018} \frac{\sin(x)}{x} dx = 0.$$

附件 C

A useful characterization of integrability

Theorem ♣: A function $f(x)$ on $[a, b]$ is integrable if and only if for any $\epsilon > 0$, there exists a partial P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Proof of Sufficiency:

Proof of Necessity:

Application: If $f(x)$ is integrable on $[1, 5]$, then it is integrable on $[2, 3]$.

附件 C

Classes of integrable functions

Definition: $M(\varphi)$, $m(\varphi)$ **and** $\omega(\varphi)$

Lemma ♣♣:

$$\sup_{x\in J, x'\in J} |\varphi(x) - \varphi(x')| = \omega(\varphi).$$

Proof: Quiz.

$\omega_i(f)$**:**

$\Omega(f, P)$**:**

Re-write Theorem ♣ in terms of $\Omega(f, P)$**:** `(This will be important!)`

Exercise: Compare $\omega(f)$ and $\omega(|f|)$.

Hint: Use Lemma ♣♣.

Fact: If f is integrable on $[a, b]$, then so is $|f|$.

附件 C

Two applications of ω_i and Ω:

A. If f, g are integrable on $[a, b]$, then so is fg.

Hint: Estimate $\omega_i(fg)$.

B. If f is integrable on $[a, b]$, and $m(f) > 0$, then $1/f$ is integrable on $[a, b]$.

Hint: Estimate $\omega_i(1/f)$.

附件 C

Lemma-C: A continuous function on a closed interval is integrable.

Hint: Estimate $\Omega(f, P)$.

Lemma-I: An increasing function on a closed interval is integrable.

Hint: Estimate $\Omega(f, P)$.

FTC revisited (p. 84)

$R[a,b]$:

Theorem 1 (p. 85) Let $f \in R[a,b]$, and introduce

$$\Phi(x) = \int_a^x f(t)dt.$$

Then $\Phi \in C[a,b]$.

Proof. It is indeed easy. The point is to be flexible with notations:

$$\Phi(x) - \Phi(y) =$$

Theorem 2 (p. 85) Let $f \in R[a,b]$, and and $x_0 \in (a,b)$. If $f(x)$ is continuous at x_0, then

$$\Phi(x) = \int_a^x f(t)dt$$

is differentiable at x_0 and

$$\Phi'(x_0) = f(x_0).$$

Proof. A nice example of using def and $\epsilon - \delta$.

Def of continuity:

Def of $\Phi'(x_0)$:

附件 C

A favorite type of problems: Let

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt.$$

Then

$$F'(x) =$$

Practice A: Find f' for $f(x) = \int_0^{x^2} e^{-t^2} dt$.

Practice B: Find f' for $f(x) = \int_{\sqrt{x}}^{x^2} \sqrt{1+t^2}dt$.

Practice C: Find f' for $f(x) = \int_{\ln x}^{\sin(x)} \ln(t + \tan(t))dt$.

附件 C

Three ways to approximate an definite integral:

Set-up: The point is to choose the samples ξ!

A: Rectangle R_n.

B: Trapezoid T_n.

C: Parabola S_n. This is hard to guess

$$S_n = \frac{2}{3}R_n + \frac{1}{3}T_n.$$

Task: Write it in a form similar to a Riemann sum.

Example: $f(x) = x^2$ on $[0, 4]$ with a general n.

附件 C

Three ways to approximate an definite integral: An example

Example: $f(x) = x^2$ on $[0, 4]$ with a general $n = 4$.

A: Rectangle R_n.

B: Trapezoid T_n.

C: Simpson S_n.

附件 C

Strategy for the parabola approximation (also called Simpson's formula):

Over each $I_n =$, we choose a parabola to pass the two endpoints and the middle point. What does this mean?

Now the very beautiful calculation:

$$\int_{x_{i-1}}^{x_i} (\lambda_i x^2 + \mu_i x + \nu_i) dx = \qquad ??? \qquad = \frac{1}{6}\Big[f(x_i) + 4f(\frac{x_i + x_{i-1}}{2}) + f(x_{i-1})\Big]\Delta x_i.$$

附件 C

HW: (Due on December 25) (Recall that you need to do at least four problems from below.)

1. Theorem 5 in p. 82 of the textbook.
2. Show two of the three definitions of integrals are equivalent.
3. If $f(x)$ is integrable on $[1, 5]$, then so is it on $[2, 4]$.
4. State and prove FTC.
5. Let $f \in R[a, b]$ and $f \geq 0.1$. Show that $\ln(f) \in R[a, b]$. (Q: What is $R[a, b]$?)
6. Write a one-page thesis on $\int \frac{\sin(x)}{x} dx$.
7. Complete the three approximations of $f(x) = x^2$ on $[0, 4]$ with a general n. Then decide which is the best.

After having done at least four problems above, choose any problems on or after p. 79 in the problem book.

附件 C

Error estimate for R_n (Theorem 1, p. 102)

1. The error term over $I_i =$

2. Introduce

$$c = \qquad\qquad \text{and} \quad h = \qquad\qquad ,$$

and

$$\psi(u) = \int_{c-u}^{c+u} f(t)dt.$$

3. Calculate
   - $\psi(0) =$
   - $\psi'(0) =$
   - $\psi''(0) =$
   - $\psi(h) =$
   - $\psi'(h) =$

4. Now a key formula:

The error term over $I_i =$

5. Taylor, with the integral remainder:

6. Now happy estimates:

附件 C

Wallis formula for π (p. 105)

In 1655 J. Wallis deduce the following formula for π:

$$\prod_{n=1}^{\infty}\Big(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\Big)=\frac{\pi}{2}. \qquad (\clubsuit)$$

This can be easily deduced from the Euler's product formula:

$$\frac{\sin(x)}{x}=\prod_{n=1}^{\infty}\Big(1-\frac{x^2}{n^2\pi^2}\Big). \qquad (\clubsuit\clubsuit)$$

Another obvious payoff of the Euler formula is

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\ \clubsuit\clubsuit\clubsuit\ =$$

Now your chance to find it by yourself!!!

(Hint for left: Taylor expand $\sin(x)$; hint for right: multiply it out!)

Unfortunately, we won't prove Euler's formula now, but we can still prove Wallis' formula.

We introduce

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n(x)dx. \qquad (\clubsuit\clubsuit\clubsuit\clubsuit)$$

Then we find its recursive formula by ???

附件 C

Now we have

- $I_0 =$
- $I_1 =$
- $I_{2n} =$
- $I_{2n+1} =$

Notation: $n!!$

Now we can rewrite I_n succinctly as

Now we write down the consequence of

$$\sin^{2n+1}(x) < \sin^{2n}(x) < \sin^{2n-1}(x).$$

This will squeeze out the conclusion!

附件 C

Friendly version of the Stirling formula (p. 107)

It is said that the following is familiar:

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1} \tag{✠}$$

Assume so, then we proceed to prove the friendly version of the Stirling formula:

$$\sqrt[n]{n!} \sim \frac{n}{e}. \tag{♣}$$

Here $A_n \sim B_n$ if $\lim_{n\to\infty} \frac{A_n}{B_n} = 1$.

> The Stirling formula:
> $$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n.$$

附件 C

The Stirling formula:

Easy start: Use just one rectangle and one trapezoid to estimate the integral

$$\int_n^{n+1} \frac{1}{x}\, dx.$$

We can re-write the above easy estimate as

$$0 < \big(n + \frac{1}{2}\big) \ln \big(1 + \frac{1}{n}\big) - 1 < \frac{1}{4}\Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Introduce $a_n = \frac{n!}{\sqrt{n}\cdot\left(\frac{n}{e}\right)^n}$ and we want $\lim a_n = a = ???$

$\ln \frac{a_n}{a_{n+1}} =$

Now we have

$$1 < \frac{a_n}{a_{n+1}} < ?$$

Summary so far:

- a_n is
- $a_n e^{-\frac{1}{4n}}$ is
- $\lim_{n\to\infty} a_n - a_n e^{-\frac{1}{4n}} =$

♣ Happy nest intervals:

附件 C

There are many ways to evaluate the number a and we use

$$\frac{\pi}{2} = \lim \frac{1}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \tag{$\clubsuit$}$$

`Now your fun to complete one of the most famous formulas in mathematics!`

Generalized Integrals and Improper Integrals

A. $\int_0^\infty \frac{1}{1+x^2}dx$.

It is "generalized" because we only defined integrals on closed intervals like $[a, b]$.

It is a small operation to extend it to $[0, \infty)$:

$$\int_0^\infty \frac{1}{1+x^2}dx \doteq \lim_{A\to\infty} \int_0^A \frac{1}{1+x^2}dx =$$

Another method: Change of variable by $x = \tan(t)$**:**

B. $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$

It is "improper" because we only defined integrals for bounded functions.

It is a small operation to extend it to $[0, 1]$ or $(0, 1)$ or ??? :

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx \doteq \lim_{\eta\to 0^+} \int_\eta^1 \frac{1}{\sqrt{1-x^2}}dx =$$

Another method: Change of variable by $x =$???

附件 C

C. (HW) $\int_1^{\int} e^{-ax}\sin(bx)dx$**, where** $a > 0$**.**

E. (HW) $\int_0^1 x\ln(x)dx$

D. $\int_0^1 \ln(x)dx$ We try two ways:

(a) by definition

(b) pretend that it is integrable.

(Important) Consider $\int_a^\infty \frac{1}{x^p}dx$.

Q: What are reasonable ranges of a and p?

Now we assume $a = 1$ and $p > 0$.

(Still important) Consider $\int_0^a \frac{1}{x^p}dx$.

Now we assume $a = 1$ and $p > 0$.

附件 C

Convergence test for generalized integrals

Consider

$$\int_0^\infty f(x)dx$$

and define

$$\Phi(H) = \int_0^H f(x)dx.$$

Practice: The $\epsilon - \delta$ formulation of the convergence of $\int_0^\infty f(x)dx$.

Practice: The Cauchy formulation of the convergence of $\int_0^\infty f(x)dx$.

The test:

- Given $f(x), g(x)$ on $[a, \infty)$ for some $a \in \mathbb{R}$.
- Suppose that both are integrable for all $[a, H]$, where $H > a$.
- Moreover, for some (large) Δ, we have

$$|f(x)| \leq g(x), \qquad x \in [\Delta, \infty).$$

- Then the convergence of

$$\int_0^\infty g(x)dx$$

  implies the convergence of

$$\int_0^\infty f(x)dx$$

Examples/HW: Analyze the convegence of $\int_0^\infty \frac{\sin(x)}{x}dx$ and $\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx$

Proof.

附件 C

The first mean value theorem for integration: For some $c \in [a,b]$, we have

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Q: What are the conditions for f, g?

- For $f \in C[a,b]$
- For $g \in C[a,b]$ and $g \geq 0$

Q: Given an example to show that the condition $g \geq 0$ cannot be dropped.

Proof.

Application: Show that $\int_2^\infty \frac{1}{x \ln x} dx$ is divergent. (Q: What about $\int_1^\infty \frac{1}{x \ln x} dx$?)

Hint: Use the Cauchy condition. (Q: What is it?)

附件 C

The second mean value theorem for integration: For some $c \in [a, b]$, we have

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Good practice: What are the conditions for f, g?

- For f: increasing/decreasing and ???
- For g:

Proof by IBP:

Example: Show that $\int_0^\infty \frac{\sin(x)}{x}$ is convergent. (**Hint:** Cauchy.)

附件 C

The Dirichlet test: Let $f, g \in C[a, \infty)$.

Assume that f is decreasing, with

$$\lim_{x\to\infty} f(x) = 0.$$

Moreover, we assume that there is a $K \geq 0$ such that

$$\Big| \int_a^H g(x)dx \Big| \leq K, \qquad \forall H \geq a.$$

Then we claim that the integral

$$\int_a^\infty f(x)g(x)dx$$

is convergent.

Q: What is a good example?

The improper integral version:

Proof.

附件 C

Example: $\int_1^\infty \frac{1}{x(\ln x)^p}dx$ (Try a few different methods)

Example 9: Study the convergence and absolute convergence of

$$\int_0^1 \frac{\sin\frac{1}{x}}{x^p}dx \qquad (0 < p \leq 2).$$

Hint: Divide into many cases.

附件 C

Lemma (p. 50) Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real number sequences.

Suppose that there exists $N_0 \in \mathbb{N}$ such that

$$|\alpha_n| \leq \beta_n, \quad \forall n > N_0.$$

If $\{\beta_n\}$ is infinitesimal, then so is $\{\alpha_n\}$.

Lemma (p. 50) If $\{\alpha_n\}$ is infinitesimal, then it is also bounded.

附件 C

Theorem (p. 51):

(A) If both $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal, then so is $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$.

(B) If $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal and $\{\beta_n\}_{n=1}^{\infty}$ is bounded, then $\{\alpha_n\beta_n\}_{n=1}^{\infty}$ is infinitesimal.

Lemma (p. 50) Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real number sequences.

Suppose that there exists $N_0 \in \mathbb{N}$ such that

$$|\alpha_n| \le \beta_n, \quad \forall n > N_0.$$

If $\{\beta_n\}$ is infinitesimal, then so is $\{\alpha_n\}$.

Lemma (p. 50) If $\{\alpha_n\}$ is infinitesimal, then it is also bounded.

附件 C

Theorem (p. 51):

(A) If both $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal, then so is $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$.

(B) If $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal and $\{\beta_n\}_{n=1}^{\infty}$ is bounded, then $\{\alpha_n\beta_n\}_{n=1}^{\infty}$ is infinitesimal.

1. Give two ways to say that a sequence is not infinitesimal; then prove that they are equivalent.

Proof.

---

2. If $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, then so is $\beta_n = \frac{\alpha_1+\cdots+\alpha_n}{n}, \quad n = 1, 2, \cdots$.

附件 C

3. Show that if $\lim x_n = a$, then $\lim \sin(x_n) = \sin(a)$.

---

4. Show that if $\lim x_n = a$, then $\lim \sqrt[3]{x_n} = \sqrt[3]{a}$.

---

5. Show that if $\lim x_n = a, \lim y_n = b$, then $\lim x_n y_n = ab$.

1. State and prove the Cauchy convergence principle.

2. State and prove the nested interval theorem.

3. What is B-W? State and prove the B-W theorem.

---

4. Give an example of a non-convergent sequence, and use the Cauchy criteria to justify your example.

1. State and prove the two basic lemmas from page 104.

♣

♣ ♣

---

2. First write a complete statement for

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{A}.$$

Then prove it by $\epsilon - \delta$.

附件 C

3. Use two methods to show that

$$\sup_{x\in D}\{-f(x)\} = -\inf_{x\in D} f(x),$$

here $f(x)$ is defined on D.

---

4. Use $\epsilon - \delta$ to define the nine possibilities of $\lim_{x\to\Delta} f(x) = \#$.

附件 C

Quiz 6 (2018-10-18)   Name(Print) ____________________

1. Suppose that $f(x)$ is continuous at $x = 0$, and for any $x, y \in (-\infty, \infty)$, we have

$$f(x+y) = f(x) + f(y).$$

Show that $f(x) = cx$ for some c.

---

2. Let $f(x) \in [a, b]$, and assume that $|f(x)|$ is monotone. Show that $f(x)$ is monotone. Is the converse true?

附件 C

3. Let $f(x) \in C(-\infty, \infty)$, and suppose that

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

exist. Then prove that $f(x)$ is u.c.

---

4. Use function transform $y = \tan z$ to simplify the equation

$$y'' = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2.$$

附件 C

Quiz 7 (2018-10-25) Name(Print) ______________________________

1. State the two definitions of uniform continuity and prove their equivalence.

2. State and prove the uniform continuity theorem.

3. Is $y = \sqrt[3]{x}$ uniformly continuous on $[0, \infty)$? Justify your answer please.

---

4. True or false: If $f(x)$ is uniformly continuous on $(0, 1)$, then $f(x)$ is bounded on $(0, 1)$. Justify your answer.

附件 C

Quiz 8 (2018-11-01) Name(Print) ____________________

1. State and prove the first important theorem for $C[a, b]$.

---

2. State and prove the second important theorem for $C[a, b]$.

附件 C

3. State and prove the third important theorem for $C[a,b]$.

---

4. State and prove the fourth important theorem for $C[a,b]$.

Quiz 9 (2018-11-08) Name(Print) ____________________

1. Suppose that $f(x)$ is continuous at $x=0$, and for any $x, y \in (-\infty, \infty)$, we have

$$f(x+y) = f(x) + f(y).$$

Show that $f(x) = cx$ for some c.

---

2. Let $f(x) \in C(-\infty, \infty)$, and suppose that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist. Then prove that $f(x)$ is u.c.

附件 C

3. Let $f(x) \in [a, b]$, and assume that $|f(x)|$ is monotone. Show that $f(x)$ is monotone. Is the converse true?

---

4. Use function transform $y = \tan z$ to simplify the equation

$$y'' = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2.$$

---

5. Use variable transform $x = \tan t$ to simplify the equation

$$y'' + \frac{2x}{1+x^2}y' + \frac{y}{(1+x^2)^2} = 0.$$

附件 C

1. Is $f(x) = \sqrt{x}\ln(x)$ uniformly continous on $(0, \infty)$? Please explain why.

---

2. Is $g(x) = x\ln(x)$ uniformly continous on $(0, \infty)$? Please explain why.

3. Let $f(x) \in C(a,b)$ and $x_0 \in (a,b)$. Assume that we know that $f'(x)$ exists on $(a,b) \setminus \{x_0\}$, and we know that $\lim_{x \to x_0} f'(x) = A$ for some $A \in \mathbb{R}$. Then show that $f'(x_0)$ exists and is indeed equal to A.

---

4. Assume that $f(x)$ is differentiable on $\mathbb{R}$ and assume that $\lim_{x \to \infty} = \infty$. Show that $f(x)$ is not uniformly continuous.

---

5. Assume that $f(x)$ is differentiable on $(0,a)$ and $\lim_{x \to 0^+} \sqrt{x} f'(x)$ exists. Show that $f(x)$ is u.c. on $(0,a)$.

附件 C

1. Show that the following function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{1}$$

has derivatives of any order on $\mathbb{R}$.

Streamelined proof of the L'Hopital's rule for $\frac{\infty}{\infty}$ (Version 1): Let $a, L \in \mathbb{R}$.

- Let $f(x), g(x)$ be defined on some (punctured) neighborhood $U(a, \eta)$ of a, such that

$$\lim_{x \to a} g(x) = \infty.$$

- Then $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ implies $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

---

a. **[Proof.]** We claim that it is sufficient to assume $L = 0$.

b. So our goal now is to show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

c. For any $\epsilon > 0$, choose $\delta_1 > 0$ such that

$$\left| \frac{f'(\xi)}{g'(\xi)} \right| < \epsilon, \quad \forall 0 < |\xi - a| < \delta_1.$$

d. Now we choose an auxiliary point

$$x_0 = a + \frac{\delta_1}{2}.$$

e. Pause and draw a pic please!

f. Observe that

$$\lim_{x \to a} \frac{g(x_0)}{g(x)} = \lim_{x \to a} \frac{f(x_0)}{g(x)} = 0,$$

because ...

so there exists a $\delta_2 > 0$ such that

$$\left| \frac{g(x_0)}{g(x)} \right| < 1 \quad \text{and} \quad \left| \frac{f(x_0)}{g(x)} \right| < \epsilon, \quad \forall 0 < |x - a| < \delta_2.$$

(Better another pic.)

---

g. Now we choose δ (please use a different one from our class)

$$\delta = ???.$$

We can use $\epsilon - \delta$ to show $\lim_{x\to a^+} \frac{f(x)}{g(x)} = 0$. Indeed, we shall show that

$$\left| \frac{f(x)}{g(x)} \right| < 3\epsilon, \qquad \forall\ a < x < a + \delta.$$

h. Now the two complicated formulas show up finally:

附件 C

Quiz 12 (2018-11-29) Name(Print) ____________________

1. Explain carefully that if $f(x)$ is u.c. on $[1,3]$ and $[2,\infty)$, then f is u.c. on $[1,\infty]$.

---

2. Prove/disprove that if $f(x)$ is u.c. on $[1,2]$ and $[2,\infty)$, then $f(x)$ is u.c. on $[1,\infty)$.

---

3. Show that if $f(x)$ is Lipschitz on $\mathbb{R}$, then f is u.c. (But you need to google what is "Lipschitz"!)

---

4. True or false, and why: If $f(x)$ on $\mathbb{R}$ satisfies the Holder condition with exponent $\frac{1}{2}$, then $f(x)$ is u.c. (But you need to google what is "Holder condition"!)

附件 C

5. Find the Maclaurin series of $\ln(\sin(x) + \cos(x))$ up to the fourth order.

---

6. Write down the Maclaurin series of

$\sin(x) \sim$

$\cos(x) \sim$

$\ln(1 + x) \sim$

$\ln(2 + x) \sim$

$(1 + x)^{\frac{\pi}{2}} \sim$

$\arctan x \sim$

附件 C

Hint: Everything is about Taylor & hidden somewhere in the problem book.

1. Show that e is irrational.

---

2. Show that the Euler constant γ is well defined.

附件 C

3. Let $f(x) \in C^2([0,1])$ and $f(0) = f(1) = 0$. Assume that $|f''(x)| \leq 1$. Show that

$$|f(x)| \leq \frac{1}{8} \qquad \forall x \in [0,1].$$

(Optional: Is the constant optimal?)

---

4. Assume that $f(x) \in C^{(2)}(\mathbb{R})$ and $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all x. Show that

$$|f'(x)| \leq \sqrt{2}, \qquad \forall x \in \mathbb{R}.$$

(Optional: Is the constant optimal?)

Quiz 14 (2018-12-20) Name(Print) ______________________________

1. State and prove FTC

---

2. A continuous function on a closed interval is integrable.

附件 C

3. An increasing function on a closed interval is integrable.

---

4. Define $\omega(\varphi)$ for a function φ on a closed interval J. Then show that

$$\sup_{x\in J, x'\in J} |\varphi(x) - \varphi(x')| = \omega(\varphi).$$

---

5. Use $\Omega(\cdot)$ to show that if f is integable on $[1, 5]$, then so is it on $[2, 3]$.

附件 C

1. State and deduce the error estimate for R_n. (Hint: Theorem 1 in page 102.)

Theorem 1.

Proof. Recall the integral remainder formula:

$$R_{n+1} =$$

---

2. Show that

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1}$$

---

3. Show the friendly version of the Stirling formula:

$$\sqrt[n]{n!} \sim \frac{n}{e}.$$

Here $A_n \sim B_n$ if $\lim_{n\to\infty} \frac{A_n}{B_n} = 1$.

附件 C

Advanced Calculus: Spring 2019

Xiang Fang
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Office Hour: Tuesday 2pm-2:50pm

- Textbook: Books of Zhang and the problem book of Fang.
- Grading: HW 30%, Quizzes 30%, Exams 40%.
- TA: Kuo Li-Feng 0926020285, d0955466322@gmail.com

(108-02-19 Tuesday)

Q: How to represent a point, or a vector in $\mathbb{R}^m$?

Q: How to represent a sequence of points in $\mathbb{R}^m$?

Q: What is the norm, then modulus, and the length of a vector $x = (x_1, \cdots, x_m)$?

What is the triangle inequality and the Cauchy inequality?

Q: Are they equivalent?

The convergence of a sequence of points in $\mathbb{R}^m$?

Notation: $U(a,\eta) =$ $\check{U}(a,\eta) =$

Theorem. Let $\{x_n\}$ be a sequence in $\mathbb{R}^m$ and $a \in \mathbb{R}^m$. Show that $x_n \to a$ iff $x_n^i \to a^i$ for each $i = 1, \cdots, m$.

Cauchy sequence in $\mathbb{R}^m$:

Cauchy convergence principle in $\mathbb{R}^m$:

Cluster point (p.149): We say that a point a is a cluster point of a set D if

---

Equivalent formulation:

Definition: We say that as x approaches a in D, the limit of the function $f(x)$ is A if

Set-up:

- Domain D:
- The target point a:
- The function f:
- A:

Draw a pic please!

The $\epsilon - \delta$ version:

附件 **D**

Example 4 (p. 151) Consider the limit of

$$f(x,y) = \frac{x^2y^2}{x^2+y^2} \qquad ((x,y) \neq (0,0))$$

as $(x,y) \to (0,0)$.

1. Sequential

---

2. $\epsilon - \delta$

Example 5 (p. 152) Consider the limit of

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad ((x,y) \neq (0,0))$$

as $(x,y) \to (0,0)$.

Example 6 (p. 152) Consider the limit of

$$f(x,y) = \frac{x^2y}{x^4+y^2} \qquad ((x,y) \neq (0,0))$$

as $(x,y) \to (0,0)$.

附件 **D**

Definition of continuity: Sequential version

- Domain D:
- Target point a:
- Function f:

Definition: We say that $f(x)$ is continous at a in D if

Definition of continuity: $\epsilon - \delta$ **version**

Isolated point:

Q: What does it mean if a point a is in D but not a cluster point of D?

Claim: A function is automatically continuous at any of its isolated points.

Practice: Find the domain of $f(x, y) = \arcsin \frac{y}{x}$ and determine its continuity at each point in the domain.

Practice: Find the domain of $f(x, y) = \ln(x + y)$ and determine its continuity at each point in the domain.

State and prove the Bolzano-Weierstrass theorem in $\mathbb{R}^n$

HW I: (Due on February 26) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from p. 139 to p. 156 at least twice and I read it carefully."
2. State and prove the Cauchy convergence principle in $\mathbb{R}^m$.
3. Any problem from 15.1.1 to 15.2.17 in the problem book.

附件 **D**

Closed set:

Open set:

Another definition of open set:

Proof of equivalence:

Theorem 2 (p. 158) State and prove the boundedness theorem in $\mathbb{R}^m$.

Statement:

Proof. It is indeed similar to that in $\mathbb{R}$ and the method is by ???

Recall the definition of sup**:**

Recall the definition of inf**:**

Theorem 3 (p. 158) State and prove the max-min theorem in $\mathbb{R}^m$.

Statement:

Proof.

1. Let $M = \sup_{x \in K} f(x)$; then by the definition of sup, we can find

2. Now Bolzano-Weierstrass:

3. Now closed set:

4. Now continuity:

附件 **D**

Recall the definition of uniform continuity:

Another equivalent definition of UC:

Proof of equivalence:

Theorem 4 (p. 159) State and prove the UC theorem in $\mathbb{R}^m$.

Statement:

Proof.

1. Proof by contradiction. This means that there exists some ???

2. Now Bolzano-Weierstrass:

3. Now continuity:

附件 **D**

The general defintion of norm on $\mathbb{R}^m$:

Q: Given a point x, or a vector x in $\mathbb{R}^m$, what is the correct definition of the length of x?

A: The familiar $||x|| =$ is just one way to do it. It is not the unique way.

Three other ways to define the length of x:

-
-
-

The general definition of a norm on $\mathbb{R}^m$:

- It is a function on $\mathbb{R}^m$, that is,
- it is positive, that is,
- it is scaling invariant, that is,
- it is reasonable, that is,

Now we repeat and give the rigorous definition.

Definition:

We say that two norms are equivalent if

From norm to metric:

附件 **D**

Theorem 1 (p. 165) Equivalent norms determine the same convergence.

Reformulation in symbols:

Proof:

Theorem 1 again (p. 165) Equivalent norms determine the same continuity.

Reformulation in symbols:

Proof:

Example or counter-example:

Claim: Any norm on $\mathbb{R}^m$ is continuous w.r.t. the Euclidean norm.

Reformulation:

Proof:

Claim: The Euclidean sphere is a closed set. (?)
Proof:

Theorem 2 (important, p. 166) Any two norms on $\mathbb{R}^m$ are equivalent.

Proof.

1. Max-min for $N(\cdot)$ on the sphere:

2. Now please fill in the rest of the proof by yourself!

HW II: (Due on March 12) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 156-179 at least twice and I read it carefully."
2. **True or false:** Any two metrics on $\mathbb{R}^n$ are equivalent?
3. State and prove Theorem 2 in p. 174.
4. Any problem from 15.1.1 to 15.3.8 in the problem book.

Cartesian product $E \times F$**:** (After Rene Descartes (1596-1650) ???)

- `Latinized form of the name of French philosopher and mathematician Ren Descartes.`

The educated way to describe it:

Recall the definition of a metric space

Set-up:

Three axioms:

-
-
-

Exercise: Draw two different examples of "unit ball" in $\mathbb{R}^2$ other than the familiar one!

So when we write $U(a, \eta)$, strictly speaking, we should write

$$U^d(a, \eta).$$

Example 1:

Example 2:

附件 D

True or false: Any two metrics on $\mathbb{R}^n$ are equivalent???

Fundamental definitions in a metric space (X, d)

Interior points: (Similarly, exterior points)

Boundary points:

Q: Is a boundary point p of the set E in the set E?

Partition: $E =$

Closed sets:

Q: How about X and $\emptyset$?

---

open sets:

Q: How about X and $\emptyset$?

Theorem 2 (p. 174) The complement of an open set is a closed set; the complement of a closed set is an open set.

Remark: The proof is similar to that in $\mathbb{R}^m$. So, homework and quiz.

Exercise 1: $U(a, \eta)$ is always open.

Exercise 2: The interior E^{int} is always open.

Exercise 3: The exterior E^{int} is always open.

附件 **D**

Thm 3 (p. 175) Let (X, d) be a metric space and let $E \subset X$ be any subset.

Let $\bar{E} = E \cup \partial E$, and we call it the closure of E. Then

(1) $\bar{E}$ is closed.

(2) Any point $c \in \bar{E}$ is the limit of a sequence in E. (We allow the dummy sequence here.)

(3) $\bar{E}$ is the smallest closed set containing E.

Warning: When we write the words "the smallest closed set", we are facing a danger ...

Proof. (1) This follows from the partition ...

---

(2) Two cases. First the dummy case, that is, $c \in$?

Now assume that $c \in \partial E$.

---

(3)

Exercise/homework/quiz: Show that

$$\partial E = \partial(E^c).$$

Exercise/homework/quiz:

- Open sets: Finite intersections and countable unions are still open.
- Closed sets: Finite unions and countable intersections are still closed.

Cauchy sequence in a metric space (Sometimes called "fundamental sequence".)

Easy theorem (p. 176) Any convergent sequence is Cauchy.

Important Definition: (complete metric space)

Q: Is $\mathbb{R}^m$ complete? (Warning: Don't give a quick answer!)

Write your own examples, complete or not

♣ In particular, find a space X with d_1 and d_2, such that (X, d_1) is complete but (X, d_2) is not.

Banach's fixed-point theorem (Also known also the contraction mapping principle)

Theorem: Any contractive mapping on a complete metric space has a unique fixed point.

Contractive:

Exercise: Any contractive mapping is continuous.

Fixed point:

The proof of uniqueness is easy!

Proof of existence:

1. Start with any $x_0 \in X$. We get a sequence x_n by iteration:

2. x_n and x_{n+1} are very close to each other!

3. Claim: $\{x_n\}$ is Cauchy.

4. Wrap up:

附件 **D**

Applications: Picard-Lindelof Theorem in ODE

Theorem: Consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Assume that f is defined in a neighborhood of $(t_0, .y_0)$ and is uniformly Lipschitz in y and continuous in t.

Then there exists an $\epsilon > 0$ such that there exists a unique solution $y(t)$ to the initial value problem on the interval

$$[t_0 - \epsilon, t_0 + \epsilon].$$

---

1. Assume that f is defined on

$$C_{a,b} = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$$

and let

$$M = \sup_{C_{a,b}} |f|.$$

2. We introduce X by

$$X = C\Bigg([t_0 - a, t_0 + a], [y_0 - b, y_0 + b]\Bigg),$$

with the sup norm.

3. X is complete. This is hard, and depends on uniform continuity, and is not really very hard.
4. Let L be the Lipschitz constant of f w.r.t. the second variable.

This means that for any t, y, y' in domain, we have

$$|f(t, y) - f(t, y')| \leq L|y - y'|.$$

5. Here we choose (how?)

$$a < \min\left\{\frac{b}{M}, \frac{1}{L}\right\}.$$

6. We define a (contractive) mapping

$$\Gamma : X \to X$$

by

$$\Gamma(\varphi)(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

7. The hard work is to justify the above sentence.
8. Now the easy part: Apply Banach.

附件 **D**

HW III: (Due on March 19) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 179-195 at least twice and I read it carefully."
2. Show that the p-norm on $\mathbb{R}^m$ satisfies the triangle inequality iff $p \geq 1$.
3. Let $U \subset \mathbb{R}^m$ be a bounded open set such that $0 \in U$. Show that one can find a norm $N(\cdot)$ on $\mathbb{R}^m$ such that U is the unit ball under N iff U is convex.
4. Let $E \subset [0, 1]$ be the collections all x such that its decimal expansion contains only the digits 1 and 3. Then find the closure of E.
5. Show that any open subset of $\mathbb{R}^1$ is the union of at most countably many disjoint open intervals.
6. Any problem from 15.1.1 to 15.3.8 in the problem book.

Definition 1 (p. 179) Sequential compactness of $K \subset X$:

Most Important Examples:

Open Cover of $E \subset X$:

Definition 4 (p. 180) Compactness of $C \subset X$:

Important Theorem: In $\mathbb{R}^m$, the following three classes of subsets are the same:

(1) Compact subsets;

(2) Sequential compact subsets;

(3) Bounded closed subsets.

Rectangles in $\mathbb{R}^m$ with $l(I) =$

- Closed:

- Other:

Observation (Lemma 1 in p. 181): How to bisect a rectangle in $\mathbb{R}^m$?

Draw a pic!

Lemma 2 (Nested rectangle theore, p. 182) Please formulate and prove by yourself!

Proof of the important theorem. Now please recall it so you can get familiar with it.

(1)

(2)

(3)

---

(1) $\to$ (2)

Given: K and $\{x_n\}$.

Want: $a \in K$ s.t.

By contradiction:

(2) → (3) By contradiction again.

The easy part:

The still easy part:

---

(3) → (1) The key direction, by contradiction again.

1. Suppose $K \subset I_0$ and there exists $\mathcal{V}$ s.t.

2. Bisect to get I_1:

3. Iterating:

4. The nested rectangle theorem:

5. **Claim:** $c \in K$.

6. Observation: $c \in V \in \mathcal{V}$.

Now some easy geometry:

Draw $c \in V \in \mathcal{V}$ first.

Theorem 2 (pp. 185-186) Let (X, d) be a metric space, let $K \subset X$ be compact, and let $f \in C(K)$.

(1) f is bounded.

Hint: (a) We produce an open set around each point $a \in K$!
(b) For each open set in the cover, we bound f.

Theorem 2 (pp. 185-186) Let (X, d) be a metric space, let $K \subset X$ be compact, and let $f \in C(K)$.

(2) f is uniformly continuous.

Proof of (2).

(a) For any $\epsilon > 0$ and any $b \in K$, there exists an $\eta > 0$ such that for any $x, x' \in U(b, \eta)$, we have

$$|f(x) - f(x')| < \epsilon.$$

(b) Now we form an open cover of K by

$$\left\{ U(b, \frac{\eta}{2}) \right\}_{b \in K}.$$

(c) Let $\delta = \min\{\frac{\eta_1}{2}, \cdots, \frac{\eta_q}{2}\}$.

Claim: This δ will do the job. (Q: what job?)

(d) For any $x, x' \in K$ with $d(x, x') < \delta$, ...

`Draw a pic to illustrate the situation!`

附件 D

Lebesgue number δ of $(K, \mathcal{V})$. Set-up: Let (X, d) be a metric space, and let $K \subset X$ be sequentially compact., and let $\mathcal{V} = \{V\}$ be an open cover of K.

Lemma/Claim: There is a number δ such that if a subset $S \subset X$ satisfies

- $S \cap K \neq \emptyset$, and $\text{diam}(S) < \delta$,

then $S \subset V$ for some $V \in \mathcal{V}$.

Proof by contradiction:

(i) If the above δ does not exist, then there exists a sequence of subsets

$$S_n \subset X, \qquad n = 1, 2, \cdots,$$

such that

(ii) To use the sequential compactness, we choose x_n ...

(iii) Now we get a limit point

$$\lim_{k\to\infty} x_{n_k} = a.$$

(iv) For this a, we cover it!

Theorem (p. 188) For any metric space, compactness is the same as sequential compactness.

Proof.

1. We only need to assume that K is sequentially compact.
2. Now to show compactness, we assume that we are given an open cover $\mathcal{V} = \{V\}$.
3. Let δ be the Lebesgue number of $(K, \mathcal{V})$ and let
$$\epsilon = \frac{K}{2}.$$
4. Claim: It is sufficient to show that
$$\mathcal{U} = \{U(x, \epsilon)\}_{x \in K}$$
admits a finite cover of K.
5. Now argue by contradiction. That is, we assume ...
6. Clearly, pick any $x_1 \in K$ and $U(x_1, \epsilon)$ cannot cover K. So we can find $x_2 \in K \setminus U(x_1, \epsilon)$.
7. Then $U(x_1, \epsilon)$ and $U(x_2, \epsilon)$ cannot cover K. So ...

Chapter 12. Section 1. Partial derivative and total differential

Set-up: $f(x, y)$ defined in a neighborhood of (x_0, y_0).

Purpose: To study the change of f near that point. How?

A: We study the change along a direction.

So, what is a direction? Let $e =$

$$L: \quad x = \qquad\qquad\qquad , y =$$

What does it mean if we travel a distance t along this direction?

What is the change of f along this traveling?

Definition: The directional derivative of f at (x_0, y_0) along e:

Partial derivatives:

Example Calculate the partial derivatives of $f(x, y) = x^2y + e^{x+y}$.

Example Calculate $\frac{\partial f}{\partial e}$ for any e:

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

附件 **D**

Total differential: Recall the one variable case, and then try to define $df(x_0, y_0) =$

Differentiable What is this for one variable?

Another way to write $o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$.

Theorem 1. Differentiability implies continuity.

Re-state:

Proof.

Theorem 2. Differentiability implies the existence of directional derivatives.

Re-state:

Proof.

Summary: The formula for the directional derivative.

Theorem 3. Suppose that f has partial derivatives in a neighborhood of (x_0, y_0).
Suppose that these partial derivatives are continuous at (x_0, y_0).
Then f is differentiable at (x_0, y_0). In particular, it is continuous at the point.

Proof.

1. Consider $f(x_0 + h, y_0 + k) - f(x_0, y_0) =$

2. Now we use one-variable calculus:

3. Now continuity of partial derivatives:

4. Consider $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ again

5. Now we recall the definition of being differentiable, and then we check that we are done.

$C^0(\Omega)$:

$C^1(\Omega)$:

Copy the diagram from the book here!

Composition (pp. 206-207)

Consider $F =$

Set-up:

Important strategy: We find the total differential, then the derivative formulas follow.

(a) The differential formula of f:

(b) Plug φ^i into above

(c) The differential formula of φ^i:

(d) $F(t) - F(t_0) =$

(e) The end is here:

The general case:

附件 **D**

Challenging Example (p. 211): Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Important observation: To get $f_{xy}(0,0)$, we need to find first

$$f_x(0,y)$$

But this one requires us to treat two cases separately!

♣ $f_x(0,y)$ with $y \neq 0$

♣ $f_x(0,0)$

Now $f_{xy}(0,0)$

附件 D

Important Theorem (p. 212). Suppose $f(x, y)$ is defined in a neighborhood of (x_0, y_0).
Suppose f_{xy} and f_{yx} exist in a neighborhood of (x_0, y_0).
Suppose that f_{xy} and f_{yx} are continuous at (x_0, y_0).
Then we have

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Proof.

1. Introduce
$$\varphi(x) = f(x, y_0 + k) - f(x, y_0)$$
and
$$\psi(y) = f(x_0 + h, y) - f(x_0, y).$$

2. Claim:
$$\varphi(x_0 + h) - \varphi(x_0) = \psi(y_0 + k) - \psi(y_0).$$

3. The left by one variable calculus, twice!

4. The right by one variable calculus, twice!

5. Happy merging:

6. Continuity of f_{xy} and f_{yx} concludes the proof.

附件 **D**

Notations: (a, b) **and** $[a, b]$ **where** $a, b \in \mathbb{R}^m$

Q: What is MVT and what is the m**-dimensional version of MVT?**

Target: $f(b) - f(a)$

Strategy: Write it as $\varphi(1) - \varphi(0)$ so we can apply MVT.

Theorem 2 (p. 221)

Set-up: We use $C^1(D)$!

Easy application: Recall that "a function is constant iff its derivative is zero." Now, please formulate and prove its m-dimensional version.

Domain:

Function:

附件 **D**

(p. 226) Implicit differentiation: Let $F(x, y) = 0$ on some domain and this determines y as a function of x: $y = y(x)$. Now we need to find $\frac{dy}{dx}$.

This is not hard by the chain rule:

Formula:

Q: Why $y = y(x)$? exists (and is differentiable)?

♣ **This is an important and difficult theorem in theory.**

Definition:

1. Let $D \subset \mathbb{R}$ and $E \subset \mathbb{R}$ be open intervals.
2. Let $F(x, y)$ be defined on $D \times E$.
3. If for each $x \in D$, there exists a unique $y \in E$ such that

$$F(x, y) = 0,$$

then we say that F **determines an implicit function** $y = f(x)$ **from** D **to** E.

Example: $x^2 + y^2 = 1$. Then what is D and E?

(p. 228) The implicit function theorem

1. Let $(x_0, y_0) \in \Omega$ be a point in an open set $\Omega \subset \mathbb{R}^2$.
2. Assume that $F \in C^1(\Omega)$.
3. Assume that
$$F(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$
4. Then there exists an open rectangle
$$D \times E \subset \Omega$$
with (x_0, y_0) as the center, such that
   - For any $x \in D$, there exists a unique $y \in E$ such
   $$F(x, y) = 0.$$
   So we obtain a function $y = f(x)$ from D to E.
   - $y = f(x) \in C^1(D)$ and
   - the derivative is given by the formula
   $$\frac{dy}{dx} = \qquad\qquad\qquad\qquad .$$

Proof. `This is long, so be patient please. Regard it as a test of your tenacity.`

1. We assume that on a rectangle
$$[x_0 - \gamma, x_0 + \gamma] \times [y_0 - \eta, y_0 + \eta] \subset \Omega$$
where $\gamma, \eta > 0$, we have
$$\frac{\partial F}{\partial y}(x, y) > 0.$$

2. We claim that
$$F(x_0, y_0 - \eta) < 0 < F(x_0, y_0 + \eta).$$
We show this by considering a new function in y:
$$\psi(y) = F(x_0, y).$$
Then $\psi'(y) = ...$

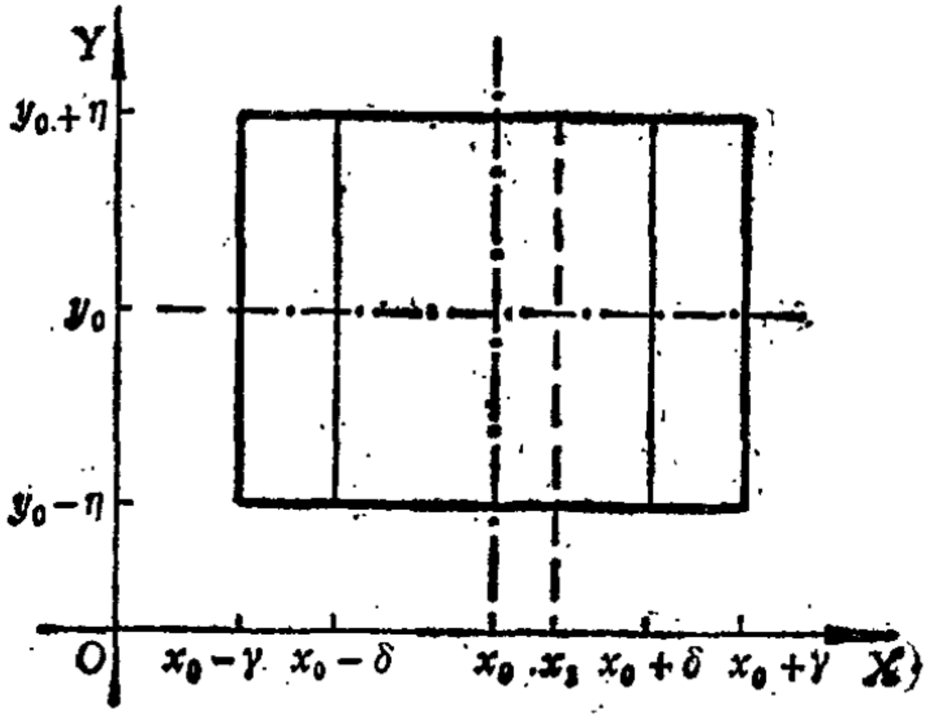
附件 D

3. We claim that we can find $\delta \in (0, \gamma)$ s.t.

$$F(x, y_0 - \eta) < 0 < F(x, y_0 + \eta), \qquad \forall x \in (x_0 - \delta, x_0 + \delta).$$

We show this by considering two new functions in x:

$$F(x, y_0 - \eta) \quad \text{and} \quad F(x, y_0 + \eta).$$



4. We claim that the following choice of D and E work: *(Q: What does it mean?)*

$$D = (x_0 - \delta, x_0 + \delta) \quad \text{and} \quad E = (y_0 - \eta, y_0 + \eta).$$

5. Our next task is to show that for any $x_1 \in D$, the equation

$$F(x_1, y) = 0 \qquad (♣1)$$

has a unique solution in E.

6. To show (♣1) we consider a function in y

$$F(x_1, y)$$

附件 D

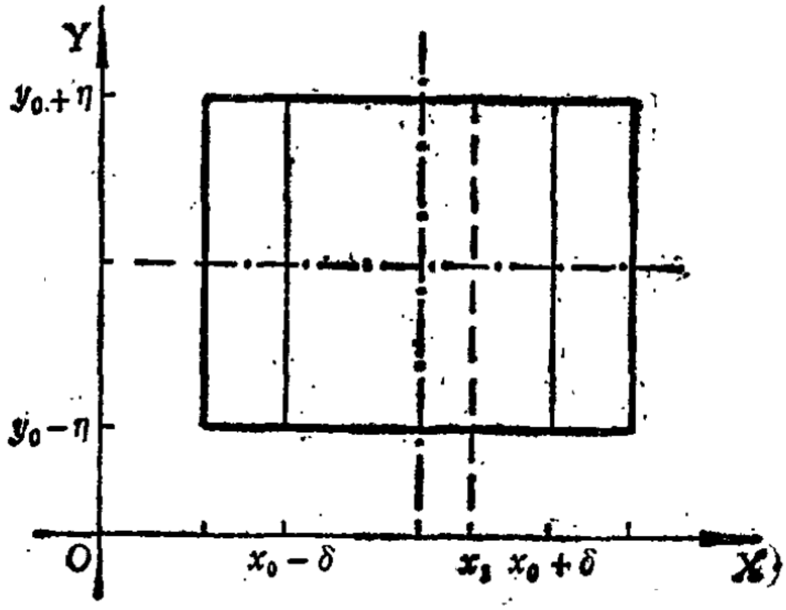
7. Now we have the function f from D to E:

$$y = f(x) \quad \text{such that} \quad F(x, f(x)) = 0.$$

8. Our next task is to show that

$$f \in C(D) \tag{$\clubsuit$2}$$

9. To show ($\clubsuit$2), we look at any $x_1 \in D$ and let $y_1 = f(x_1)$.



10. **Want:** For any small $\epsilon > 0$, there exists $\sigma > 0$ s.t.

11. We can (trivially) assume

$$(y_0 - \epsilon, y_0 + \epsilon) \subset (y_0 - \eta, y_0 + \eta).$$

Then we claim that

- $$F(x_1, y_1 - \epsilon) < 0 < F(x_1, y_1 + \epsilon).$$

- There exists $\sigma > 0$ such that

$$F(x, y_1 - \epsilon) < 0 < F(x, y_1 + \epsilon), \qquad x \in (x_1 - \sigma, x_1 + \sigma).$$

- Now we observe that for any $x \in (x_1 - \sigma, x_1 + \sigma)$, we have

$$f(x) \in (y_1, -\epsilon, y_1 + \epsilon).$$

This proves ($\clubsuit$2).

12. Our next task is to show that

$$\text{“}f(x) \text{ is differentiable on } D.\text{”} \qquad (\clubsuit 3)$$

That is,

13. For convenience, for any $x \in D$ and $h \in \mathbb{R}$ with small $|h|$, we denote

$$k = f(x+h) - f(x).$$

14. Consider the two variable MVT for

$$? = F(x+h, \qquad\qquad) - F(x, y) =$$

15. Now we have

$$\frac{k}{h} =$$

16. Happy ending with $h \to 0$.

HW VI: (Due on April 9) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 226-238 at least twice and I read it carefully."
2. (Optional) State that "I have read the textbook from pp. 239-262 at least twice and I read it carefully." Note that this part requires much linear algebra.
3. State that "I have read the textbook from pp. 263-268 at least twice and I read it carefully."
4. Any problems from the handout on 03-21.
5. Any problems from HW V.
6. Give the necessary definitions to show that l^2 is a Hilbert space.
7. Give the necessary definitions to show that $C[0,1]$ is a Banach space. (Hint: You may use theorems on uniform convergence directly. We will prove them later. Yes, use them first!)
8. Any problem from 16.1.1 to 17.6 in the problem book.

Give a complete statement for the implicit function theorem that $F(x_1, x_2, \cdots, x_n) = 0$ **determines a function** $x_n = f(x_1, \cdots, x_{n-1})$. In particular, give a formula for

$$\frac{\partial x_n}{\partial x_i} = \qquad\qquad\qquad\qquad .$$

Set-up: That is, the domain and the function.

Assumption:

Conclusion and formula:

Re-write: $F(x_1, \cdots, x_n, y) = 0$.

Another situation: What is the implicit function theorem for

$$F_1(x, y_1, y_2) = 0, \qquad F_2(x, y_1, y_2) = 0 \ ?$$

This is a beautiful combination of calculus and linear algebra.

Hint: Find the formula for derivatives first, and then you can guess what is the condition.

Now you can guess a complete statement:

Exercise: Generalize it further (Theorem 3, p. 236) and further (Theorem 4, p. 237).

`The key is the concept of Jacobian (p. 236):`

$$\frac{\partial(F^1, \cdots, F^p)}{\partial(y^1, \cdots, y^p)} \doteq$$

(A related notion is $D(F)$.)

§ 9 Inverse Function Theorem (p. 263)

Definition (Homeomorphism)

C^1-homeomorphism:

Local homeomorphism:

Theorem 1 (Inverse Function Theorem/Inverse Mapping Theorem)

- Let $\Omega \subset \mathbb{R}^m$ be open and let $a \in \Omega$.
- Let $f \in C^1(\Omega, \mathbb{R}^m)$.
- If $\det Df(a) \neq 0$, then f is locally C^1-homeomorphic.

Here Df is just a shorthand for the Jacobi matrix, or Jacobian for short.

In other words, if $f(a) = b$, then there exists $a \in U$ and $b \in V$ s.t.

Proof.

i. Consider a function
$$F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$$
defined by
$$F(x, y) = f(x) - y.$$
Why bother? Check the implicit function theorem. But note that we want to solve out x.

附件 D

ii. By the implicit function theorem, we claim that there exists an open rectangle $W \subset \Omega$ with center a and an open rectangle $V \subset \mathbb{R}^m$ with center b, such that **(pic?)**

(1) For any $y \in V$,

(2) The map $g : V \to W \subset \mathbb{R}^m$ is and

$$Dg(y) =$$

---

iii. It is sufficient to show that $g : V \to g(V)$ is a homeomorphism.

iv. Consider the set

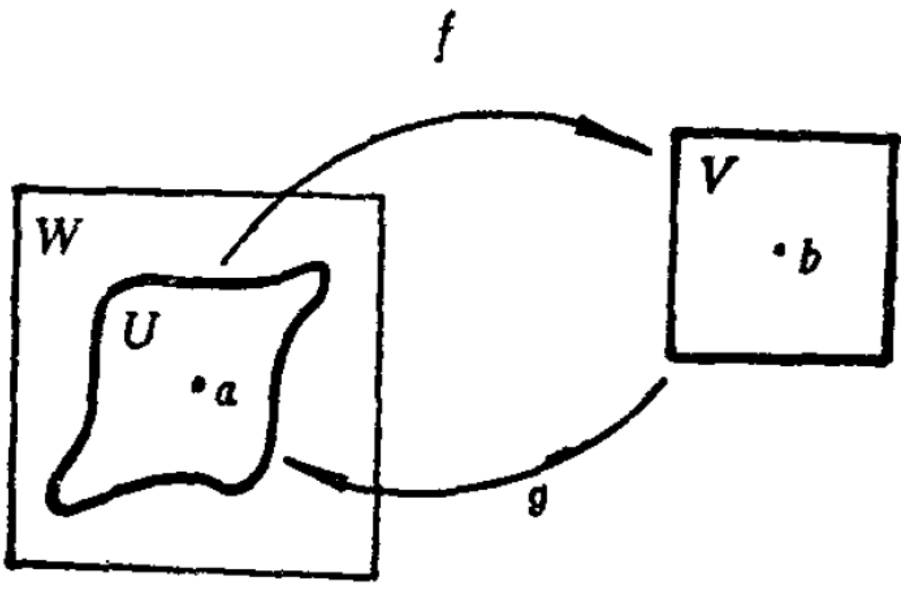
$$g(V) \doteq U = \{x \in W \ : \ f(x) \in V\}.$$

It is sufficient to show that U is open. To show this, we look at any $x_0 \in U$,

- by the continuity of f at x_0, we can find $\epsilon > 0$, and then $\delta > 0$, such that

(1)

(2)



HW VII: (Due on April 16) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 263-284 at least twice and I read it carefully."
2. State and prove the implicit function theorem.
3. State and prove the inverse function theorem.
4. State and prove the Sylverster theorem.
5. Give an example of a function in three variables with a local max at $(1, 2, 3)$ and you need to verify it by checking that the Hessian is negative definite.
6. Graph the level curves of $F(x, y) = x^3 + y^3 - 3xy$.
7. Any problem starting from p. 221 of the problem book.

(§ 10 Extreme Values of Multi-variable Functions (p. 268)

Local max/min:

Theorem 1 (p. 269) (Easy first derivative test)

Set-up:

Conclusion: If f has an extreme value at a, then we necessarily have

Proof.

Critical points:

Next in order to derive sufficient conditions, we need to expand $f(x) = f(x_1, \cdots, x_m)$ at a point $a = (a_1, \cdots, a_m)$ up to the second order in the Taylor expansion.

Challenge: Expand $f(a + h) = f(a)+$???

Quadratic forms: $Q(\xi) = \sum c_{ij}\xi_i\xi_j$

Positive definite:

Example:

Another example:

Sylverster Theorem:

Proof for $m = 2$**:**

Lemma (p. 273) Positivity implies lower bound. (Q: Isn't it trivial?)

Statement:

Proof.

Theorem 2 (p. 274) The second derivative (Hessian) being positive implies local min.

Set-up:

Claim:

Proof. $f(a+h) - f(a) =$

This page includes some small practices on level curves of $F(x, y) = 0$:

Q: How to find the tangent of a level curve?

Q: How to find the tangent of a curve?

Q: How to graph the gradient of $F(x, y)$?

Exercise: Graph one level curve for each of $F(x, y) = x^2 + y^2$ and $G(x, y) = xy$ such that they are tangent to each other.

Then find the normal lines at the tangent point.

Lagrange multiplier Example: Find the maximal value of

$$f(x, y) = xy$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 4 = 0.$$

Naive method: Pick a value of f and try to see whether it is a max!!!

For example, pick 10 and ask ourself the question:

Q: Is $f = 10$ the maximum value? Why?

Q: Is $f = 1$ the maximum value? Why?

The general problem (Version 1) $f = f(x, y)$

The general problem (Version 2) $f = f(x) = f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+p})$.

A bare-hand solution (p. 276): We try to solve $x_{m+1}, \cdots, x_{m+p}$ from the p equations

$$g_i(x) = 0, \qquad i = 1, \cdots, p.$$

Example: $f = xy$ with $g = x^2 + y^2 = 4$.

The general case (p. 276):

Target:

Constrain:

KEY: Conditions to solve $x_{m+1}, \cdots, x_{m+p}$

Summary so far: We now have a free extreme value problem.

附件 **D**

The practical version of Lagrange multiplier: Theorem 3 with $p =$? (p. 277)

- Given a target function $f(x) = f(x_1, \cdots, x_m, x_{m+1})$
- Given a constraint $g(x) = g(x_1, \cdots, x_m, x_{m+1}) = c$.
- Assume that f achieves an extreme value at $a = (a_1, \cdots, a_{m+1})$
- Then there exists $\lambda \in \mathbb{R}$, called the lagrange multiplier, such that

$$\nabla f(a) = \lambda \nabla g(a). \qquad (\clubsuit)$$

In other words, this equation (♣) will tell us the candidates for extreme points.

Important observation: We have $m + 1$ equations to solve $m + 1$ unknown variables.

Proof by picture:

Proof by calculus:

A discussion of what assumptions we need to put:

Important strategy: If we want the formula

$$\nabla f(a) = \lambda \nabla g(a)$$

to be meaningful, then we need to ask

Q: Is this condition sufficient? That is, can we solve x_{m+1}?

Copy Theorem 3 in p. 277 and Theorem 4 in p. 280 here. Then prove them in HW/quiz.

Example 2 (p. 281) Find the largest area among all triangles with perimeter $2p$. (Hint: Heron)

No Lagrange:

Lagrange:

Example 3 (p. 282) Find the max of $f(x_1, \cdots, x_n) = x_1 \cdots x_n$, where $x_1, \cdots, x_n$ are non-negative numbers such that $x_1 + \cdots + x_n = C > 0$.

Example 4 (p. 283) Let A be a (symmetric) matrix. Find the max and min of $f(x) = \langle Ax,\ x\rangle$ over the unit sphere.

HW VIII: (Due on April 23) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 285-307 at least twice and I read it carefully." (But this time it is ok if you skip the proofs.)
2. State definition 1 of integrals in p. 290 and Corollary 1 in p. 301 of the book.
3. Prove the $1+1$ case of Theorem 1 in p. 299.
4. Example 5 in p. 329.
5. Example 6 in p. 330.
6. Any problem starting from p. 221 of the problem book.

(p. 287) Closed Rectangle in $\mathbb{R}^m$:

Partition $P = \{P^1, \cdots, P^m\}$

Sample points $\xi = \{\xi_J\}$:

Riemann sum:

Defintion 1 ($\epsilon - \delta$, p. 290)

Iterated Integral (p. 298)

Rectangle in $\mathbb{R}^{n+p}$:

$$Q = V \times W$$

Three ways to integrate over $f(x, y)$ over Q:

附件 D

Corollary 1 (p. 301)

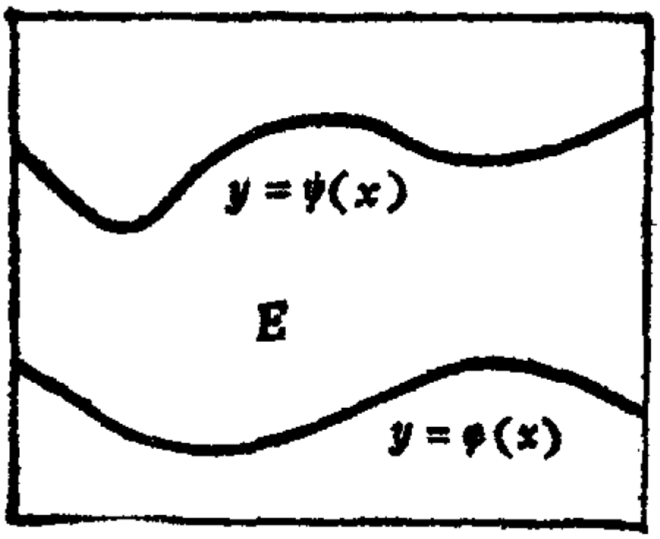
- Assume that a $n+p$-variable function $f(x,y)$ is integrable over a closed cube $Q = V \times W$.
- Assume that for each $x \in V$, $f(x,y)$, as a function in y, is integrable over W.
- Then

$$\int_Q f(x,y)d(x,y) = \int_V \left(\int_W f(x,y)dy \right) dx.$$

- An important special case is when $f \in C(Q)$, then we have

---

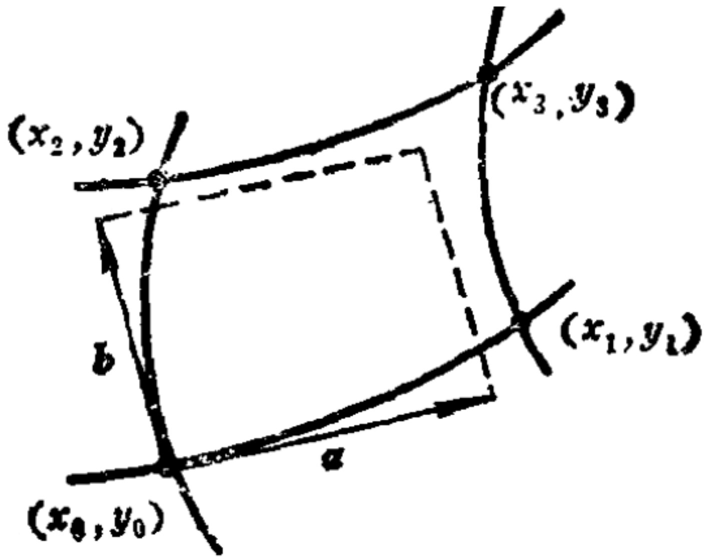
Re-state the above special case for $\mathbb{R}^m$:



附件 D

The secret of Jacobian (pp. 340-341): $dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv$

1. Interpretation: Given $\psi : D \to E$ we can integrate to get
$$\int_D f(x,y)dxdy = \int_E f(*,*)\frac{\partial(x,y)}{\partial(u,v)}dudv.$$
2. It is sufficient to consider $f = 1$.
3. It is sufficient to consider E to be a small uv-rectangle.
4. What is a small uv-rectangle?
5. Q: What is D if E is a small uv-rectangle?



6. What are the four vertices of E?
7. What is the linear approximation a?
   - $x_1 - x_0 \approx y_1 - y_0 \approx$

   Now $a =$???
8. Now $b =$???
9. area(E) $\approx |a \times b| =$???

附件 **D**

Quiz 09 for May 02: Two problems randomly chosen from 18.4.5- 18.4.9 in the problem book. Plus Example 6 in pp. 330-331. of the textbook. So three problems in total.

Q: Where does the r come from in the formula

$$dxdy = rdrd\theta \ ?$$

1. What is a polar rectangle?

2. What is its area?

3. What is the corresponding region in the xy-plane?

4. What is the corresponding area in the xy-plane?

5. Now, why does it tell us that

$$dxdy = rdrd\theta \ ?$$

A magic calculation:

HW X: (Due on May 7) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 1-16 at least twice and I read it carefully."
2. State that "I have read the textbook from pp. 32-48 at least twice and I read it carefully."
3. Prove $\kappa = \frac{||r' \times r''||}{||r'||^3}$ directly (without using N and B).
4. Any problem starting from 18.1.1 to 18.2.7 of the problem book.
5. Any problem starting from pp. 277 - 288 of the problem book.

(p. 4) Tangent line of a curve in $\mathbb{R}^3$:

Three ways to represent a curve in $\mathbb{R}^3$:

1. Parametric

---

2. Cartesian

---

3. Implicit

(p. 8) Tangent plane of a surface in $\mathbb{R}^3$:

Three ways to represent a surface in $\mathbb{R}^3$:

1. Parametric

---

2. Cartesian

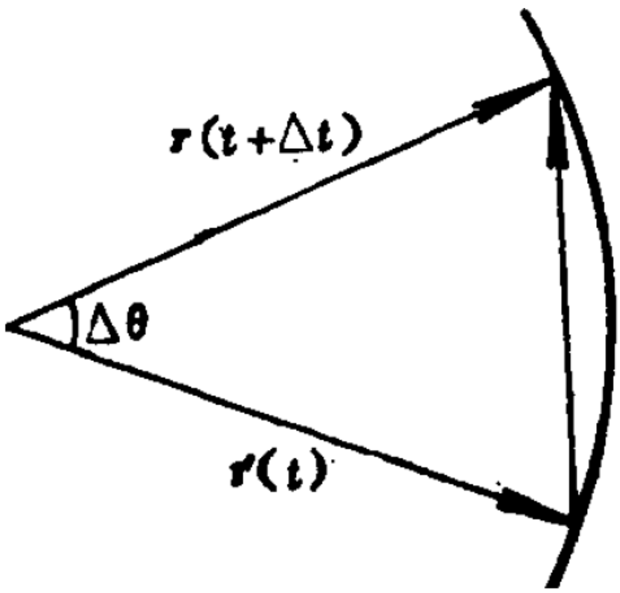
---

3. Implicit

Lemma 1 (p. 11) Let $r_1(t)$ and $r_2(t)$ be two vector-valued functions. What is the product rule for differentiation?

Lemma 2 (p. 11) Let $r_1(t)$ a vector-valued function. Then we apply the above lemma to study when it has constant length?

Lemma 3 (p. 12) Now we study the geometric meaning of $||r'(t)||$ when $r(t)$ is on the unit sphere. (Guess you answer first, please!)

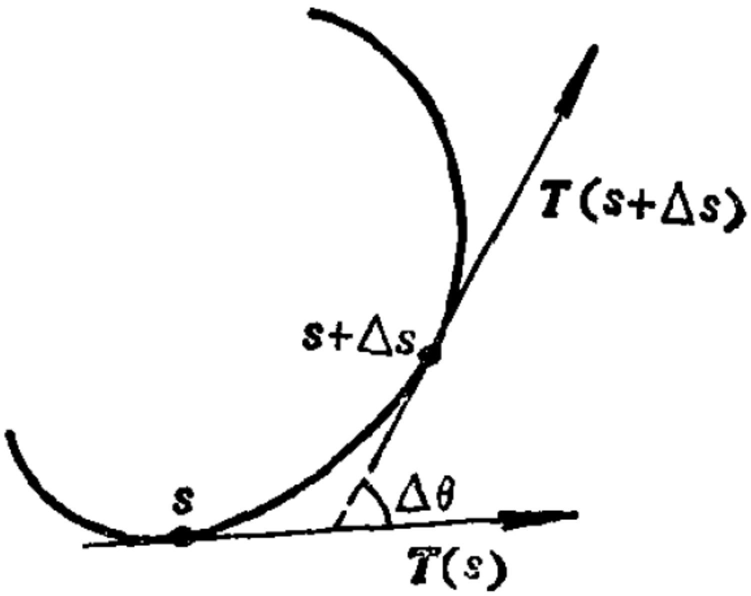


Arc length parameter (p. 13)

Derivative w.r.t. ds

$$T = T(s) = \dot{r}(s).$$

What is the meaning of $||\dot{T}(s)||$**?**



Curvature $\kappa(s)$ **and its reciprocal:** (With another formula in HW!)

附件 D

Example 1 (p. 15) Find the curvature of a circle.

Example 2 Show that a curve is a straight line iff its curvature is zero.

Practice: Show that the sum of the length of the three segments cut by the tangent plane of $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ on the three axes is a constant.

Line integral: the first kind

Q: How to find the mass of an un-even metal line?

The standard procedure of calculus:

$$\Delta m_j =$$

The definition of the line integral: the first kind

The practical formula: From ds to dt.

Line integral: the second kind

Q: How to integrate a vector field along a curve, and why bother?

`Q: How to find the work done by a force along a curve?`

Line integrals of the second type (p. ??)

Set-up:

- A vector field, such as ...
- A moving particle, such as ...

Want: The work done by the force.

Standard calculus business: This means we cut ...

The work done by the force during the j^{th} small segment is

$$W_j =$$

where

$$\Delta r_j =$$

Riemann sum:

Important: In terms of components instead of the inner product

---

Definition of the second type line integral:

$$\int_\gamma F \cdot dr = \int_\gamma Pdx + Qdy + Rdz,$$

where

In parametric form: That is, ...

附件 **D**

Two important examples:

A: Find $\int_C x dy$ over a circle.

Find $\int_C y dx$ over a circle.

A trick:

Practice: Do something similar for a triangle!

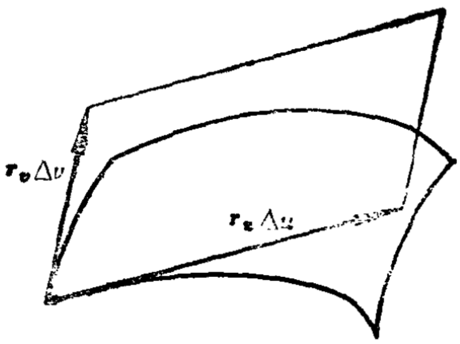
B:Find $\int_C \frac{xdy - ydx}{x^2+y^2}$ over the unit circle.

附件 **D**

HW XI: (Due on May 14) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 1-16 at least twice and I read it carefully."
2. State that "I have read the textbook from pp. 39-48 at least twice and I read it carefully."
3. State and prove how to calculate the area of a parallelogram by the cross product of vectors.
4. Show the identity: $||r_u \times r_v||^2 + (r_u, r_v)^2 = ||r_u||^2||r_v||^2$.
5. Find the area of a sphere and the volume of a ball.
6. Any problem starting from 18.1.1 to 18.2.7 of the problem book.
7. Any problem starting from pp. 277 - 288 of the problem book.
8. Any problem starting from 22.2.1 to 22.3.8 of the problem book.

(p. 39) The area of a small patch on a surface:



Formula for the area of a surface:

Differential $d\sigma = W dudv$

Surface integral of the first kind:

$$\int_S f(Q) d\sigma =$$

附件 D

Formula: A, B, C form

Cartesian formula: p, q form

Example 1 (p. 44) Surface area of a sphere.

Set-up: spherical coordinates and their ???

Method I: (**Hint:** Either p, q or A, B, C)

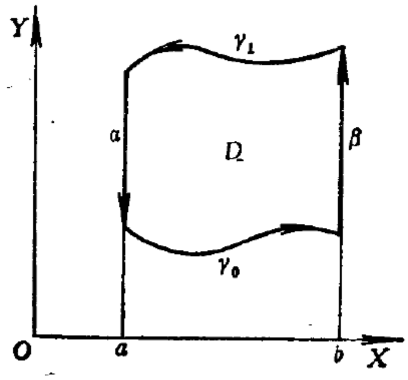
Method II: (**Hint:** Use $||r_u \times r_v||^2 + (r_u, r_v)^2 = ||r_u||^2||r_v||^2$.)

Example 2 (p. 44) Surface area of a sphere $x^2 + y^2 + z^2 = a^2$ inside of a cylinder $x^2 + y^2 = ax$.

Example 3 (p. 45) $z = xy$ inside $x^2 + y^2 = a^2$. (Q: What does it look like?)

(May 09, Thursday) § 3.a Green's formula (p. 74)

We call the following region "Type-A" and we calculate $\oint_{\partial D} P dx =$



Summary: We convert the line integral into a double integral:

Q: How to patch two type-A regions together?

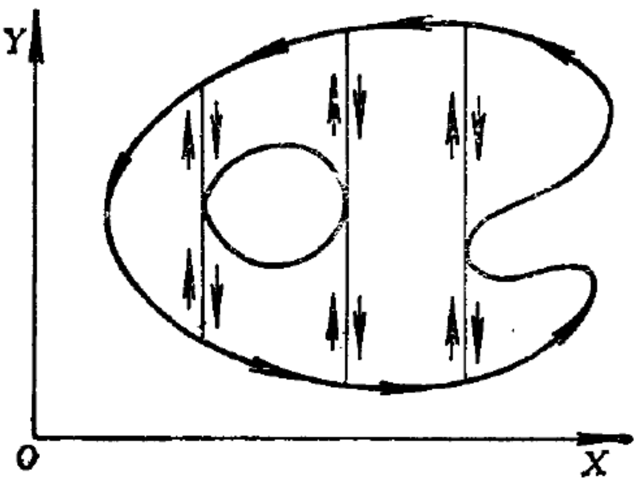
Q: How to patch a disk by type-A regions?

Practice: Draw a strange region and divide it into type-A regions!

Type-B region and $\int Qdy$:

---

non-simply connected domains: (Q: what is an educated way to say it?)



(p. 77) Green's Formula:

- Let $\Omega \subset \mathbb{R}^2$ be a bounded region such that its boundary consists of

- We specify the orietation of $\partial\Omega$ by
  - Outside:
  - Inside:
- Let $P, Q \in C^1(\bar{\Omega})$.

- Then

Example 3 (p. 80) Use Green's theorem to find the area of an ellipse.

Example 4 (p. 80) Use Green's theorem to find the area enclosed by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \qquad 0 \leq t \leq 2\pi.$$

(Q: Do you remember what is its shape?)

附件 D

Important Example (pp. 81-82) Show that

$$\int_\Gamma \frac{xdy - ydx}{x^2 + y^2}$$

is invariant with respect to any simply connected curve around the origin.

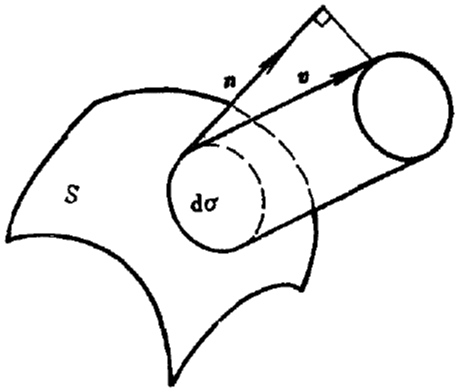
Important Example: Show that

$$\int_\gamma xdy + ydx$$

is independent of path.

(May 16, Thursday) Fluid through a surface

Surface S given by $r =$



$n =$

$v =$

$d\sigma =$

Fluid through a small patch $d\Theta =$

---

v in components:

---

n in components:

Fluid through a small patch again:

- $d\Theta =$
- $\Theta =$

Key calculuation: $\int\int_S P\cos(\alpha)d\sigma =$

附件 **D**

Summary/Definition: flux

The second type surface integral of a vector field

$$F = (P, Q, R)$$

through a surface S given by $r = r(u, v)$ is called the flux:

$$\iint_S F \cdot n d\sigma = \iint_S P dydz + Q dzdx + Q dxdy$$

Example 4:

$$I = \frac{1}{3} \iint_S xdydz + ydzdx + zdxdy,$$

where the surface S is the exterior of the sphere $x^2 + y^2 + z^2 = a^2$.

Example 5: The same integral over a rectangular box.

HW XIII: (Due on May 28) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 105-117 at least twice and I read it carefully."
2. Any problem starting from p. 285 of the problem book.

(p. 105) Independence of path:

Let $G \subset \mathbb{R}^2$ be

Let $P, Q \in C^1(G)$ be

Let $M_0, M_1 \in G$ be

Let $\gamma \subset G$ be

Definition:

Definition: We say that G is simply connected if

Theorem 1 (p. 107) Let G and P, Q be as above. Then TFAE:

1. (closed loops)

2. (indepent of paths)

3. (potential)

4. (partial derivative test)

Proof. (1) $\to$ (2). Just draw a pic!

---

(2) $\to$ (3). We define a new function

$$U(x, y) =$$

We need to check

---

(3) $\to$ (4).

---

(4) $\to$ (1). Easy if C is a simple loop. Troublesome if C has infinitely many loops.

(p. 111) Theorem 2 for multiply-connected planar doman

Let $G \subset \mathbb{R}^2$ be a domain, which means ...

Let $P, Q \in C(G)$. Then TFAE:

(a) $\int Pdx + Qdy$ is independent of paths in G;

(b) the differential form $Pdx + Qdy$ has a potential function $U(x, y)$. That is,

Proof. We only need to prove (b) $\to$ (a). (Why?)

Consider $\gamma \subset G$ to be piecewise C^1-curve in G. We can parametrize it as

Now we write $\int_\gamma Pdx + Qdy =$

(p. 114) Theorem 3 State the above theorem in $3d$.

(p. 112) Example 1 Consider again

$$\int \frac{-ydx + xdy}{x^2 + y^2} =$$

Q: What is the question?

(p. 115) Theorem 4 (But only for 2d)

Let $D \subset \mathbb{R}^2$ be a star-like domain. That is,

Let $P, Q \in C^1(D)$. Then TFAE

- $\int Pdx + Qdy$ is independent of paths.
- $Pdx + Qdy$ has a potential function.
- We have the derivative test:

Proof.

1. We only prove that (3) $\to$ (2). That is, we find a potential explicitly. (Why?)
2. Assume that D is star-like w.r.t. A. For any point $M =$ $\in D$, we define

   $$U(M) = \qquad .$$

   Want:
3. Consider any point

   $$M_0 =$$

   and its neighbor

   $$M =$$

   We claim that for small h, the triangle

   $$\Delta AM_0M \subset D.$$

4. By Green's theorem,

5. Now we have

   $$\int_{\overline{AM}} Pdx + Qdy =$$

   That is, by the definition of U,

   $$U(x_0 + h, y_0, z_0) =$$

6. We are ready to wrap up the proof:

附件 D

(Thursday, May 23, 2019) Ch. 18: Sequence and Series

Sequence:

Series:

Convergence of sequence:

Convergence of series:

Theorem 2 (p. 157) If $\sum a_n$ converges, then $\lim a_n = 0$.

Divergence test:

Example 1 (p. 157) Analyze the convergence of geometric series.

Example 2 (p. 157) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(p. 158) Convergence principle for positive series:

Example 1 (p. 158) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Example 2 (p. 159) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Theorem 1 (Baby comparison of positive series) (p. 159)

Let $\sum a_n$ and $\sum b_n$ be two positive series.

- If $\sum b_n$ converges, and

- If $\sum b_n$ diverges, and

Example 3 (p. 160) Analyze the convergence of $\sum_{n=1}^{\infty} \sin \frac{x}{n^2}$ for $x \in (0, \pi)$.

Example 4 (p. 160) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n \pm 3}}$.

附件 D

(Limit comparison theorem, p. 161) Let $\sum a_n$ and $\sum b_n$ be two positive series. If

$$\lim \frac{a_n}{b_n} = \gamma \in (0, \infty),$$

then $\sum a_n$ converges iff $\sum b_n$ converges.

Proof.

Q: What if $\gamma = 0$?

Q: What if $\gamma = \infty$?

Example 5 (p. 161) Let $x \in (0, \pi)$. Test the convergence of

(a) $\sum(1 - \cos \frac{x}{n})$

(b) $\sum 2^n \sin \frac{x}{3^n}$

Example 6 (p. 162) Analyze the following three series:

$$\sum \ln(1 + \frac{1}{n}), \quad \sum \frac{1}{n}, \quad \text{and} \quad \sum \sin(\frac{1}{n}).$$

(Cauchy's root test, version 1, p. 163) Let $\sum a_n$ be a positive series.

- If
$$\sqrt[n]{a_n} < r,$$
then $\sum a_n$ converges.
- If
$$\sqrt[n]{a_n} \geq 1,$$
then $\sum a_n$ diverges.

(Cauchy's root test, version 2) Let $\sum a_n$ be a positive series. Assume that the limit exists:

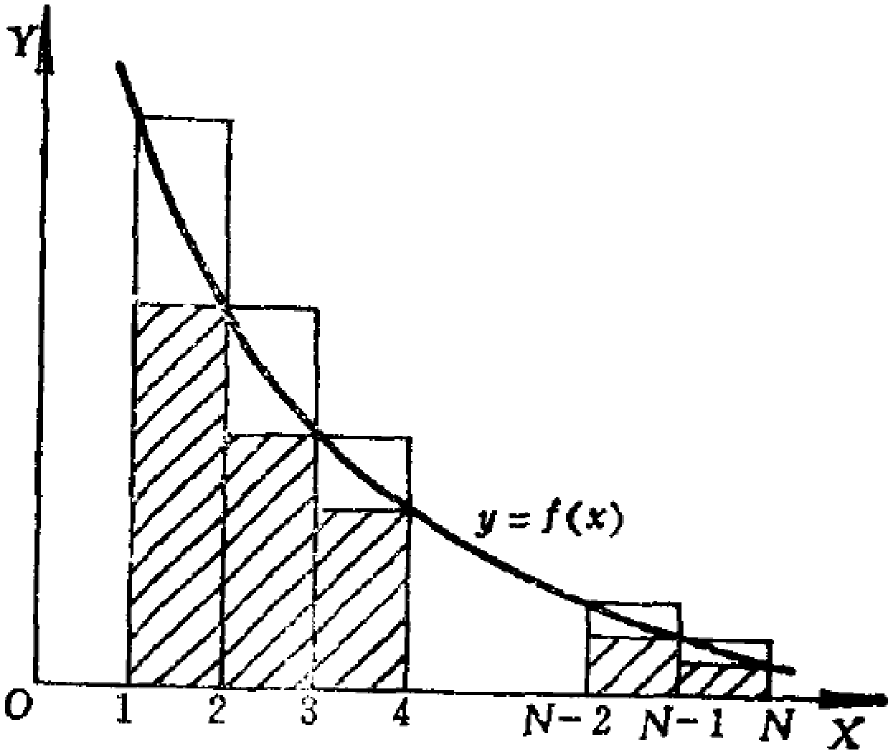
$$\sqrt[n]{a_n} = q.$$

- If $q < 1$, then $\sum a_n$ converges.
- If $q > 1$, then $\sum a_n$ diverges.

(Cauchy's integral test, p. 165) Let $f(x)$ be positive and decreasing on $[1, \infty)$. Then

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

converge or diverge at the same time.



Proof. We simply observe that

$$\sum_{n=?}^{?} f(n) \le \int_1^N f(x)dx \le \sum_{n=?}^{?} f(n).$$

Important applications (p. 167)

$\sum \frac{1}{n^p}$

$\sum \frac{1}{n(\ln n)^p}$ (How about $\sum \frac{1}{n \ln n (\ln \ln n)^p}$?)

附件 **D**

D'Alembert's ratio test: Let $\sum a_n$ be a strictly positive series. Assume that the limit exists

$$\lim \frac{a_{n+1}}{a_n} = q.$$

- If $q < 1$, then
- If $q > 1$, then

11.2.7 利用泰勒级数估计无穷小量 a_n 的阶，从而判别下列级数的收敛性：

(1) $\sum_{n=1}^{+\infty} 2^n \sin \frac{\pi}{3^n}$;　　(2) $\sum_{n=1}^{+\infty} \frac{1}{\ln(n+1)} \sin \frac{1}{n}$;

(3) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3+1}}$;　　(4) $\sum_{n=1}^{+\infty} (\sqrt{n+1}-\sqrt{n})^p \ln \frac{n-1}{n+1}$;

(5) $\sum_{n=3}^{+\infty} \ln^p \cos \frac{\pi}{n}$;　(6) $\sum_{n=1}^{+\infty} (\sqrt{n+a}-\sqrt[4]{n^2+n+b})$;

(7) $\sum_{n=1}^{+\infty} \left[e-\left(1+\frac{1}{n}\right)^n\right]^p$

11.2.8 若级数$\sum\limits_{n=1}^{+\infty}a_n$ $(a_n>0)$发散，而S_n表示级数的第n部分和. 求证：级数$\sum\limits_{n=1}^{+\infty}\frac{a_n}{S_n}$也发散.

11.2.9 若正项级数$\sum\limits_{n=1}^{+\infty}a_n$收敛，$a_{n+1}\leqslant a_n(n=1,2,\cdots)$. 求证：

$$\lim_{n\to+\infty}n\cdot a_n=0.$$

11.2.10 设

$$\begin{cases}a_n=\dfrac{1}{n^2}, & n\neq k^2, k=1,2,\cdots\\ a_{k^2}=\dfrac{1}{k^2}, & k=1,2,\cdots\end{cases}$$

求证：

(1) $\sum\limits_{n=1}^{+\infty}a_n$ 收敛；

(2) $\lim\limits_{n\to+\infty}na_n\neq 0$

11.2.11 设$0<P_1<P_2<\cdots<P_n<\cdots$. 求证：$\sum\limits_{n=1}^{+\infty}\frac{1}{P_n}$收敛的充要条件为级数

$$\sum_{n=1}^{+\infty}\frac{n}{P_1+P_2+\cdots+P_n}$$

收敛.

HW XIV: (Due on June 4) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 155-170 at least twice and I read it carefully."
2. State that "I have read the textbook from pp. 184-193 at least twice and I read it carefully."
3. Prove Theorems 3, 4, 5 in pp. 178-179 of the textbook.
4. Any problem from Ch. 11 of the problem book.

(p. 184) §4: Any series (i.e., not necessarily positive series)

Cauchy's test: This is merely a re-writing!

Example 1: $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}$

Theorem 1 (p. 184) Absolute convergence implies convergence.

Example 1 again:

附件 D

(p. 176) §3 $\limsup x_n$ & $\liminf x_n$

For any sequence $\{x_n\}$ we define two new sequences:

$$y_n = \inf_{k \geq n} x_k \quad \text{and} \quad z_n = \sup_{k \geq n} x_k.$$

Important but easy observation:

- y_n is
- z_n is

In particular,

"$\lim y_n$ and $\lim z_n$ exist if we allow infinity as a limit."

Moreover we have

- $y_n \leq x_n \leq z_n$.

It follows that

Theorem 1(p. 177) Let E be the collection of limit points of $S = \{x_n\}$.

- Show that E is closed.
- Show that

$$\liminf x_n = \inf E \quad \text{and} \quad \limsup x_n = \sup E.$$

Proof.

1. If $\xi \in E$, then

2. Now we can squeeze x_{n_k} by y_k and z_k:

   It follows that

---

3. Next we show $\liminf x_n \in E$. Then $\limsup x_n \in E$ will follow also (why?)

4. Case I: $\{x_n\}$ is not lower bounded. In this case, we can find

---

5. Case II: We assume that $\{x_n\}$ is lower bounded.
6. Claim: We shall choose two increasing subsequences

$$m_k \quad \text{and} \quad n_k$$

such that

$$y_{m_k} \leq x_{n_k} \leq y_{m_k} + \frac{1}{k}. \tag{♣}$$

7. Choose $n_0 = 0$.
8. Choose $m_1 = n_0 + 1 = 1$ and n_1 such that

9. Suppose that n_{k-1} is chose already. We choose

$$m_k = n_{k-1} + 1$$

and we choose n_k by

10. Wrap up:

Theorem 2 (p. 178) Let $\{x_n\}$ be a sequence. TFAE:

- $\liminf x_n = \limsup = \xi$;
- $\lim x_n = \xi$;
- $E = \{\xi\}$. (Q: What is E?)

Proof. We only prove (?) → (?).

Example 2 (p. 186) Let

$$\limsup \sqrt[n]{|a_n|} = q.$$

- If $q < 1$, then

- If $q > 1$, then

Example 3 (p. 186) Let

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| = q.$$

- If $q < 1$, then

- If $q > 1$, then

Let

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| = q.$$

- If $q < 1$, then

- If $q > 1$, then

附件 **D**

(108-05-30 Thursday)

Quiz 14 on June 6 will be randomly chosen from pp. 83-84 of notes.

(Lemma, p. 187) Abel summation: A powerful way to sum

$$\sum_{i=1}^{p} \alpha_i \beta_i.$$

Let

$$B_k = \sum_{i=1}^{k} \beta_i, \qquad k = 1, 2, \cdots, p.$$

$\sum_{i=1}^{p} \alpha_i \beta_i =$

Theorem 2 (Dirichlet Test, p. 189) Consider $\sum a_n b_n$. If

- a_n decreases to 0;
- B_n is bounded,

then $\sum a_n b_n$ is convergent.

Proof. We apply Abel in Cauchy:

附件 **D**

Example 7 (pp. 191-192) Let $a_n \searrow 0$. Show that

$$\sum a_n \cos nx \quad \text{and} \quad \sum a_n \sin nx$$

both converge.

Example 8 (p. 192) Let $\sum b_k$ be convergent. Show that

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} k b_k = 0.$$

Proof. We use Abel and choose ...

§5 **Properties of convergence**

Theorem 1 (p. 193, grouping) Let $\sum a_n$ be convergent. Then

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots + (a_{n_k+1} + \cdots + a_{n_{k+1}}) + \cdots$$

is also convergent.

Easy proof by considering ???

Counterexample for the converse:

Distribution: If $\sum a_n$ and $\sum b_n$ are both absolutely convergent, then so is $\sum a_n + b_n$.

Moreover, ...

Proof by considering ???

Riemann's theorem (Theorems $3 + 4$, pp. 196-198)

Let $\sum a_n$ be conditionally convergent.

Consider its re-arrangement.

Then wild things happen!

Theorem 2 (p. 194, re-ordering)

Let $\sum a_n$ be absolutely convergent.

Let a'_n be a re-ordering of $\sum a_n$. (Q: What does this mean?)

Then $\sum a'_n$ is also absolutely convergent to the same sum. That is,

---

Proof. The case of positive series is easy!

The case of arbitrary series: We use a nice trick:

$$a_n = p_n - q_n$$

where p_n, q_n satisfy

- $0 \leq p_n, q_n \leq |a_n|$
- $|a_n| = p_n + q_n$.

Task: Show that p_n and q_n are unique.

Theorem 5 (Product of two series, p. 200)

Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series.

Then $(\sum a_n)(\sum b_n)$ is defined.

Moreover,

Proof. What is a partial sum of $(\sum a_n)(\sum b_n)$?

Now we know that it is a.c., so we can arrange it according to square summation:

Example 1 (p. 201) Prove that $e^{x+y} = e^x \cdot e^y$. (Hint: You need to recall what is $y = e^x$?)

附件 D

§6 **Infinite products (p. 206)**

Definition: We say that $\prod_{n=1}^{\infty} p_n$ is convergent if

---

Theorem 1 (p. 207) The convergence/divergence test says that

Re-writing:

Examples: $\prod \left(1 - \frac{1}{n}\right)$ and $\prod \left(1 - \frac{1}{n^2}\right)$

Observation with ln **(Theorem 2, p. 208)**

Theorem 3 (p. 209) Let $a_n \geq 0$. Then $\prod_{n=1}^{\infty}(1 + a_n)$ is convergent iff $\sum a_n$ is convergent.

附件 **D**

HW XV: (Due on June 11) (Recall that you need to do at least four problems from below.)

1. State that "I have read the textbook from pp. 184-211 at least twice and I read it carefully."
2. State that "I have read the textbook from pp. 212-225 at least twice and I read it carefully."
3. Any problem from Ch. 11 of the problem book
4. Any problem from Ch. 12 of the problem book.

(p. 212) §19.1: Function sequences and series

Example 1. What is the domain of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

Example 2. What is the domain of convergence of $\sum_{n=1}^{\infty} e^{nx}$?

Example 3. What is the domain of convergence of $f_n(x) = x^n$?

Example 4. What is the domain of convergence of $\sum_{n=1}^{\infty} \frac{1}{x^n}$?

Example 5. What is the domain of convergence of $\sum_{n=1}^{\infty} (n!)x^n$?

Example 6. What is the domain of convergence of $\sum_{n=1}^{\infty} e^{nx^2}$?

(p. 214) §2. Uniform convergence: $f_n(x) \rightrightarrows f(x)$

$C(I)$**:**

Metric on $C(I)$:

$f_n(x) \to f(x)$ **on** I**:**

$f_n(x) \rightrightarrows f(x)$ **on** I**:**

Example 1 (p. 215) $f_n(x) = x^n$, but what is the domain?

附件 **D**

Example 7. (p. 213) Analyze the convergence of

$$f_n(x) = \frac{x^{2n}}{1 + x^{2n}}.$$

1. What is the domain?

---

2. Convergence

---

3. at the discontinuity points

---

4. Uniform convergence

(June 06, Thursday)

Example 2. (p. 216) Consider the sequence

$$f_n(x) = \frac{1}{n+x}.$$

Q: What is the domain?

Q: What is the limit function?

Q: Now when is it uniformly convergent?

Uniform Convergence. I $f_n(x) \rightrightarrows f(x)$

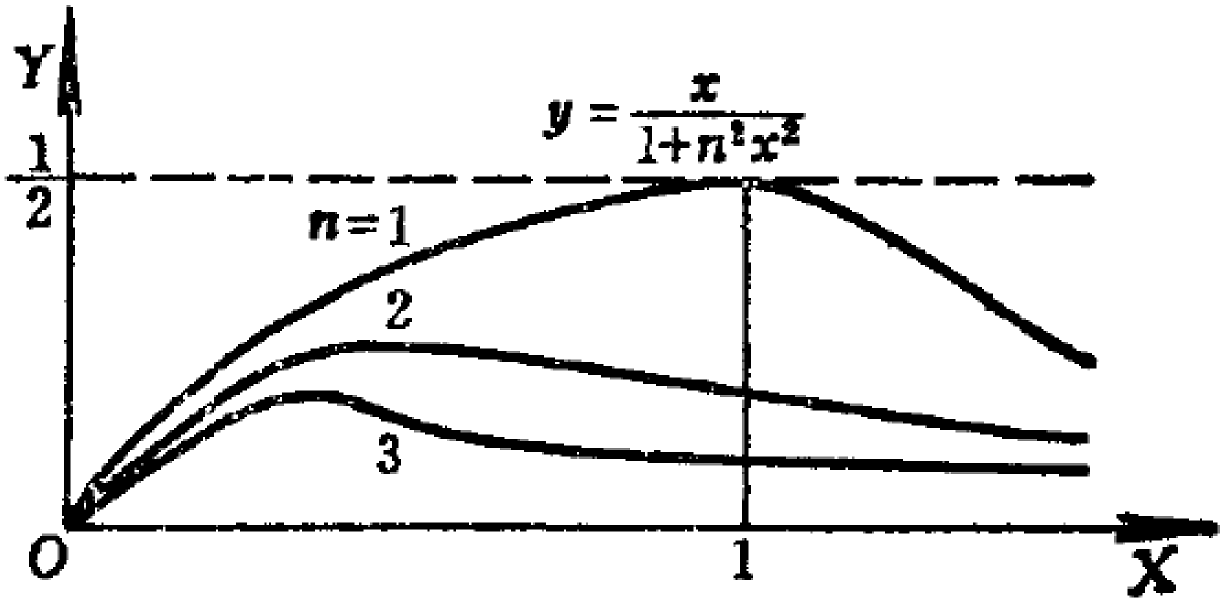
Converse:

Uniform Convergence. II $\sum u_n(x)$

附件 **D**

Example 3. (p. 218) On the interval $[0, 1]$, consider

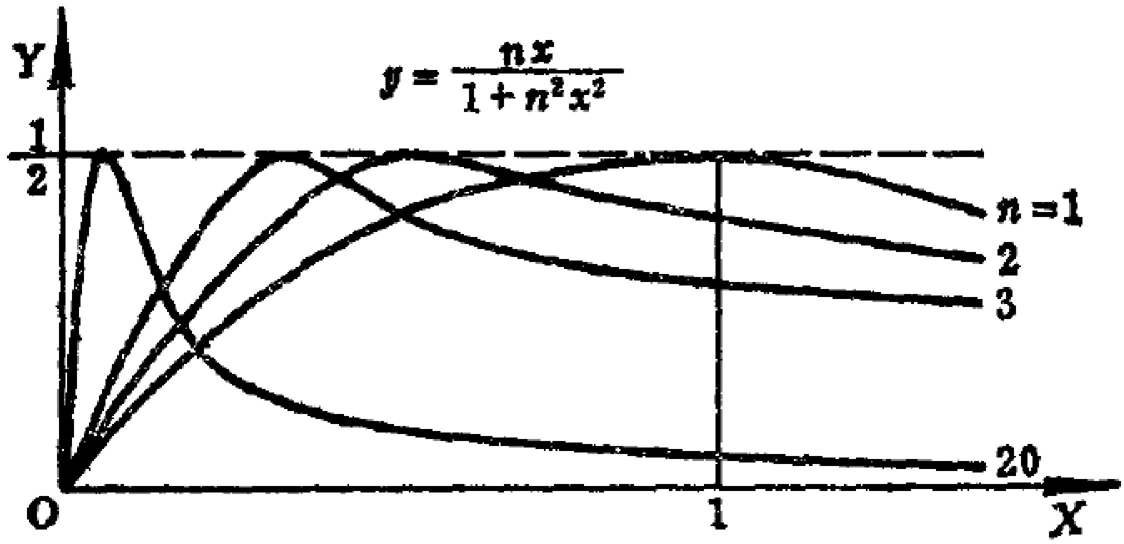
$$f_n(x) = \frac{x}{1 + n^2x^2}.$$



Q: What about other domains?

Example 4. (p. 219) On the interval $[0, 1]$, consider

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$



Q: What about other domains?

Theorem 1 (p. 219) Suppose that

$$f_n(x) \to f(x) \qquad x \in E.$$

We define

$$d(f_n, f) = \sup_{x \in E} |f_n(x) - f(x)|$$

TFAE:

(1) $f_n(x) \rightrightarrows f(x)$

(2) $\lim_{n\to\infty} d(f_n, f) = 0$

(3) For any sequence $\{x_n\} \subset E$, we have

$$\lim_{n\to\infty} (f_n(x_n) - f(x_n)) = 0.$$

Proof.

1. The equivalence between (1) and (2) are clear.
2. "(2) implies (3)" is also clear.
3. So we only prove (3) → (1) by contradiction.
4. So, if (1) does not hold, then

Example 5. Consider $f_n(x) = \frac{1}{1+nx}$. Then it is not uniformly convergent on $(0, 1)$.

Solution. By (3) in the above Theorem 1, we only need to choose

$$x_n = ?$$

Theorem 2. (Cauchy convergence principle, UC version) Indeed two versions. First seqeunce

---

Now the series version:

Theorem 3 (Weierstrass test, p. 222)

1. Suppose that $u_n(x)$ is defined on E for each n;
2. Suppose that there exists M_n such that
$$|u_n(x)| \le M_n, \qquad x \in E.$$
3. Moreover, we have
$$\sum M_n < \infty.$$
4. Then we conclude that $\sum u_n(x)$ is uniformly convergent.

Proof. We use Cauchy ...

Example 8 (p. 223) Consider

$$\sum \frac{\cos nx}{n} \quad \text{and} \quad \sum \frac{\cos nx}{n^2}.$$

12.3.2 研究下列级数在什么区间上一致收敛，其和函数在何处连续.

(1) $\sum\limits_{n=1}^{+\infty}\frac{1}{1+n^2x^2}$;　　(2) $\sum\limits_{n=1}^{+\infty}\frac{nx}{1+n^4x^2}$;

(3) $\sum\limits_{n=1}^{+\infty}\frac{\cos nx}{n^2}$;　　(4) $\sum\limits_{n=1}^{+\infty}\frac{\sin nx}{n\sqrt{n}}$;

(5) $\sum\limits_{n=1}^{+\infty}\frac{x^2}{(1+x^2)^n}$.

12.3.3 设 $f_n(x)=\frac{x^2}{x^2+(1-nx)^2}, 0\leqslant x\leqslant 1$. 求证：它的任何子序列都不一致收敛.

12.3.4 求证：$\lim\limits_{x\to 1}\sum\limits_{n=1}^{+\infty}\frac{x^n(1-x)}{n(1-x^{2n+1})}=\sum\limits_{n=1}^{+\infty}\frac{1}{n(2n+1)}$.

(06-11, Tuesday)

Recall the def of u.c.

In terms of $C(I)$:

The a_n method:

Weierstrass test:

The converse:

Example:

- Easy: $f_n(x) = \frac{1}{1+n^2x^2}$ over $[0,1]$.
- Hard: $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ over $[0,1]$.

(p. 225) Theorem 1

- Let I be an (closed) interval.
- Let $f_n(x) \in C(I)$
- Assume that $f_n \rightrightarrows f(x)$ for some $f(x)$

Then $f(x) \in C(I)$.

Proof. It is helpful to draw a picture!

1. For any $x_0 \in I$ and any $\epsilon > 0$, we hope to find δ such that ..

2. For the given $\epsilon > 0$, we first find N such that

3. For $N+1$, we look at $f_{N+1}(\cdot)$ near x_0: We find δ such that

---

An educated way to reformulate the above theorem:

Theorem 4 (Cluster point version)

- Let $E \subset \mathbb{R}$ be any set.
- Let x_0, not necessarily in E, be a cluster point of E.
- Let $f_n(x) \rightrightarrows f(x)$ on E.
- Assume that
$$a_n = \lim_{x \in E \to x_0} f_n(x) \in \mathbb{R}$$
exists!

Then

- The following limit exists
$$\lim_n a_n = a \in \mathbb{R};$$
- We have
$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x).$$

附件 D

Switch limit and integration: (Sequence version)

Theorem 2 (p. 226)

Proof. Want:

Theorem 2' (series version)

Switch limit and differentiation (Sequence version, Theorem 3, p. 227)

- Let $f_n \in C^1[a, b]$.
- Let $f_n'(x) \rightrightarrows \varphi(x)$ for some $\varphi(x)$.
- Let $f_n(x_0) \to y_0$ for some x_0, y_0.

Then we have

- $f_n(x) \rightrightarrows f(x)$ for some $f(x)$.
- $f(x) \in C^1[a, b]$ and

$$f'(x) = \varphi(x).$$

附件 **D**

§4 **Power series (p. 232)**

Theorem 1 (Cauchy-Hadamard formula) Consider the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Let

$$\rho = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

Then

- The power series is absolutely convergent for any x with

$$|x - x_0| < \rho.$$

- The power series is divergent for any x with

$$|x - x_0| > \rho.$$

Easy proof. We use test for ???

Example: Consider the convergence radii of

- $\sum \frac{x^n}{n}$
- $\sum \frac{x^n}{n^2}$
- $\sum \frac{x^n}{n(n+1)}$
- $\sum \frac{x^n}{n^{100}}$
- $\sum \frac{x^n}{n!}$

附件 **D**

§4.b (p. 234) If $\sum a_n(x-x_0)^n$ converges over a domain D, then it defines a function $f(x)$ on D. Now we study the properties of this function $f(x)$.

Lemma 1 (p. 235, easy but important)

- Let $\sum a_n x^n$ have a convergence radius ρ.
- Let $0 \leq r < \rho$.
- Then $\sum a_n x^n$ is uniformly convergent on $[-r, r]$.
- In particular, the summation, denoted by $f(x)$ is continuous on $[-r, r]$.
- Consequently, the summation $f(x)$ is continuous on $(-\rho, \rho)$.

Easy proof. Just choose ??? test!

Recall Able's summation: (and for Cauchy principle)

Lemma 2 (p. 235, Abel's second theorem)

- Let $\sum a_n x^n$ have a convergence radius ρ.
- Assume that it is convergent at $x = \rho$.
- Then the series is uniformly convergent on

$$[0, \rho].$$

- In particular, ??

Proof. Without loss of generality we assume $\rho = 1$.

1. So we assume that $\sum a_n$ is convergent.
2. We apply the Abel summation to
$$\sum_{k=n+1}^{n+p} a_k x^k.$$
3. That is,

附件 D

Theorem 2' (p. 236, easy and useful)

- Suppose that the power series
$$\sum a_n x^n$$
is convergent at a point $r > 0$.
- Then its convergence radius
$$\rho \geq r.$$
- Moreover, we can perform the integration by terms over $[0, r]$. (Q: What does this mean?)

Lemma 3 (p. 236) The formal derivative of a power series has the same convergence radius as the original power series!

Re-state:

Easy proof.

Theorem 3 (p. 237)

- Let x_0 be an interior point of the convergence domain of a power series $f(x) = \sum a_n x^n$.
- Then $f(x)$ has derivative of any order at x_0.
- Moreover, its derivative can be computed by diff the series term by term.

Easy proof. By induction, we only need to prove the statement for the first derivative.

For this, we just need to check ???

Example 4' (p. 238) Show that we can integrate

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

over ??? term by term, and the result is

Example 5 (p. 238) Show that we can integrate

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} ???$$

over ??? term by term, and the result is

Example 6 (p. 238, a different type) Show that we can differentiate

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

over ??? term by term, and the result is

Theorem 1 (p. 246, Weierstrass) Any continuous function on a closed interval $[a, b]$ can uniformly approximated by polynomials.

In other words, in terms of $C[a, b]$,

Algebra:

Show that $C[a, b]$ is an algebra!

Example:

Observation: An way to say that a subset $S \subset C[a, b]$ contains many functions is that

"S separates points."

This means

(p. 256) Stone-Weierstrass Theorem for $C(K)$: We choose $K = [a, b]$.

Assume that $S \subset C[a, b]$ satisfies

- $1 \in S$;
- S is an algebra;
- S separates points.

Then S is dense in $C[a, b]$.

That is,

That is,

附件 **D**

(4) $lx+my+nz=0$, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, 求 $f(x,y,z)=\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}$的极值.

18.4.6 $x^2+y^2+z^2\leqslant 1$, 求 $x^3+y^3+z^3-2xyz$ 的最大值和最小值.

18.4.7 求函数 $z=\dfrac{1}{2}(x^n+y^n)$在条件 $x+y=l\,(l>0, n\geqslant 1)$之下的极值,并证明: 当 $a\geqslant 0, b\geqslant 0, n\geqslant 1$ 时

$$\left(\frac{a+b}{2}\right)^n\leqslant\frac{a^n+b^n}{2}.$$

18.4.8 求 $f(x_1,x_2,\cdots,x_n)=x_1x_2\cdots x_n$ 在条件$\dfrac{1}{x_1}+\dfrac{1}{x_2}+\cdots+\dfrac{1}{x_n}=\dfrac{1}{a}\,(x_i>0, a>0)$之下的极值, 并证明: 当 $a_i>0\ (i=1,2,\cdots,n)$ 时,

$$n\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)^{-1}\leqslant(a_1a_2\cdots a_n)^{\frac{1}{n}}.$$

18.4.9 设 $x_k>0\,(k=1,2,\cdots,n)$. 证明:

(1) $p>1$ 时,

$$\sum_{k=1}^{n}x_k^p\geqslant n\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^p;$$

(2) $0<p<1$ 时,

$$\sum_{k=1}^{n}x_k^p\leqslant n\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^p.$$

18.4.10 求圆的外切三角形中面积最小者.

18.4.11 证明: 椭圆的内接三角形中,面积最大的三角形的一顶点的椭圆法线必与三角形的该顶点的对边垂直; 并求椭圆中

付件 B

面积最大的内接三角形.

18.4.12 长为 a 的铁丝切成两段，一段围成一个正方形，另一段围成一个圆. 这两段的长各为多少时，使它们所围正方形和圆形面积之和最大.

18.4.13 求抛物线 $y=x^2$ 和直线 $x-y=1$ 间的最短距离.

18.4.14 为了使渠道不致漏水，在渠道表面砌一层水泥. 根据流量的大小，其截面积就确定了. 假定要修的渠道是直的，其横断面是一等腰梯形（见图 12），问腰和底各为多少时水泥用量为最少.

图 12

18.4.15 凸四边形各边长分别为 a、b, c, d, 求最大面积者.

18.4.16 求一点 O，使与一个凸四边形的四顶点距离之和为最小.

18.4.17 证明椭球面 $ax^2+by^2+cz^2+2dxy+2exz+2fyz=1$ 的最大轴长 l 为如下方程之最大实根：

$$\begin{vmatrix} a-\frac{1}{l^2} & d & e \\ d & b-\frac{1}{l^2} & f \\ e & f & c-\frac{1}{l^2} \end{vmatrix}=0.$$

18.4.18 设 $f(x,y)\in C^{(1)}(\Omega)$，在 Ω 内有以 (ξ,η) 为中心的闭圆 S_R，$(x_0,y_0)\in\partial S_R$，它是 $f(x,y)$ 在 ∂S_R 上的最小值点. $r=\sqrt{(x-\xi)^2+(y-\eta)^2}$，$n$ 是 ∂S_R 上点 (x_0,y_0) 的单位外法向. 求证：

(1) 若 $\left.\frac{\partial f}{\partial r}\right|_{(x_0,y_0)}=0$，则对任何单位向量 l 都有

$$\left.\frac{\partial f}{\partial l}\right|_{(x_0,y_0)}=0.$$

付件 B

107-10-30

HW due on November 6:
指定作業：
從 3.5.11 - 3.5.21 中選擇 四題。
其他自由選擇題數請從
這個四頁的 handout 中選擇。

助教請挑戰（合作）完成這四頁的全部題目！

高阶导数与高阶微分

4.7.1 设 $f''(1)=0, f'(1)=1$. 求证：在 $x=1$ 点

$$\frac{d}{dx}f(x^2)=\frac{d^2}{dx^2}f^2(x).$$

4.7.2 设 $y_1=\arcsin x$, $y_2=\arccos x$. 求证 y_1、y_2 都满足方程：

$$(1-x^2)y''-xy'=0.$$

4.7.3 设 $y=(x+\sqrt{1+x^2})^m$. 求证：

$$(1+x^2)y''+xy'=m^2y.$$

4.7.4 求证：切比雪夫多项式 $T_n(x)=\frac{1}{2^{n-1}}\cos(n\arccos x)$ 满足方程

$$(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0.$$

4.7.5 求下列隐函数的二阶导数 y''：

(1) $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$ $(a>0)$;

(2) $x^3+y^3-3axy=0$ $(a>0)$.

4.7.6 求下列参数式的二阶导数 $y''(x)$：

(1) $x=a\cos^3 t, y=a\sin^3 t$ $(a>0)$;

(2) $x=a\left(\ln\operatorname{tg}\frac{t}{2}+\cos t\right), y=a\sin t$ $(a>0)$.

4.7.7 设 $y=y(x)$ 存在反函数，且满足方程

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0.$$

证明：反函数 $x=x(y)$ 满足 $\frac{d^2x}{dy^2}=1$，并由此求出一个 $y=y(x)$.

4.7.8 求下列函数的高阶导数 $y^{(n)}$：

(1) $y=\frac{1}{1-x^2}$;　　(2) $y=\frac{1+x}{\sqrt[3]{1-x}}$;

附件 D

107-10-30

(3) $y=\frac{x^n}{1-x}$;　　(4) $y=\sin^2 x$;

(5) $y=\sin^3 x$;　　(6) $y=e^x\sin x$;

(7) $y=\frac{x^n}{x^2-1}$;　　(8) $y=e^x(\sin x+\cos x)$.

4.7.9 求证:

(1) 若 $y=x^{n-1}\ln x$, 则 $y^{(n)}=\frac{(n-1)!}{x}$;

(2) 若 $y=\frac{ax+b}{cx+d}$, 则 $y^{(n)}=(-1)^n\frac{n!\,c^{n-1}}{(cx+d)^{n+1}}(bc-ad)$.

4.7.10 用数学归纳法求证:若 $y=x^{n-1}e^{\frac{1}{x}}$, 则

$$y^{(n)}=\frac{(-1)^n}{x^{n+1}}e^{\frac{1}{x}}.$$

4.7.11 设 $y=\operatorname{arc\,tg} x$. 求证:

(1) $y^{(n)}=\frac{P_{n-1}(x)}{(1+x^2)^n}$, 其中 $P_{n-1}(x)$ 为 $n-1$ 次多项式;

(2) $P_{n-1}(x)$ 的最高次项是 $(-1)^{n-1}n!\,x^{n-1}$.

4.7.12 给定函数

$$f(x)=\begin{cases}x^n\sin(\ln|x|), & x\neq 0\\ 0, & x=0\end{cases}$$

(n 为自然数). 求证: $f(x)$ 在 $x=0$ 点有直到 $n-1$ 阶的导数, 而无 n 阶导数.

4.7.13 设 $y=(\arcsin x)^2$.

(1) 证明: $(1-x^2)y''-xy'=2$,

(2) 证明: $(1-x^2)y^{(n+2)}-(2n+1)xy^{(n+1)}-n^2y^{(n)}=0$ $(n\geq 1)$;

(3) 证明: $y^{(n+2)}(0)=n^2y^{(n)}(0)$;

(4) 求 $y^{(n)}(0)$.

附件 D

107-10-30

4.7.14 设 $y=\frac{1}{\sqrt{1-x^2}}\arcsin x$.

(1) 证明: $(1-x^2)y'-xy-1=0$;

(2) 证明: $(1-x^2)y^{(n+1)}-(2n+1)xy^{(n)}-n^2y^{(n-1)}=0$ $(n\geq 1)$;

(3) 求 $y^{(n)}(0)$.

4.7.15 求证勒襄德多项式

$$P_n(x)=\frac{1}{2^n n!}[(x^2-1)^n]^{(n)} \quad (n=0,1,2\cdots)$$

满足方程:

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0.$$

提示: 令 $u=(x^2-1)^n$.

4.7.16 求证: 切比雪夫-拉盖尔多项式

$$y=e^x\frac{d^n}{dx^n}(x^ne^{-x})$$

满足方程

$$xy''-(x-1)y'+ny=0.$$

4.7.17 求证: 切比雪夫-厄尔米特多项式

$$y=(-1)^n\frac{1}{n!}e^{\frac{x^2}{2}}\frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$$

满足方程

$$y''-xy'+ny=0.$$

* * *

4.1 设 $y=\frac{x^n}{(x+1)^2(x+2)^2}$. 求 $y^{(n)}$.

4.2 设 $y=\frac{1}{\sqrt{1+x^2}}$.

(1) 求证: $y^{(n)}=\frac{P_n(x)}{(1+x^2)^{n+\frac{1}{2}}}$, $P_n(x)$ 为 n 次多项式;

附件 D

107-10-30

(2) 求证：$P_{n+1}=(1+x^2)P_n'-(2n+1)xP_n$;

(3) 求证：$P_n'=-n^2P_{n-1}$.

4.3 设 $y=e^{\sqrt{x}}$. 求证：

(1) $4xy''+2y'-y=0$;

(2) $4xy^{(n+1)}+(4n-2)y^{(n)}-y^{(n-1)}=0$;

(3) $y=\frac{d^n}{dx^n}\left[(4x)^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}y\right]$.

4.4 求证：在实轴上不存在可微函数 $f(x)$，使其满足

$$f[f(x)]=-x^3+x^2+1.$$

提示：参看 1.5 题.

4.5 求证：在实轴上不存在可微函数 $f(x)$，使其满足

$$f[f(x)]=x^2-3x+3.$$

提示：参看1.6题.

4.6 用函数变换化简方程：

(1) $y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$，令 $y=\frac{3}{\sqrt{x}}$;

(2) $y''=1+\frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$，令 $y=\mathrm{tg}\,z$.

4.7 用自变量变换化简方程：

(1) $(1-x^2)y''-xy'+a^2y=0$，令 $x=\sin t$;

(2) $x^4y''+a^2y=0$，令 $x=\frac{1}{t}$;

(3) $y''+\frac{2x}{1+x^2}y'+\frac{y}{(1+x^2)^2}=0$，令 $x=\mathrm{tg}\,t$.

4.8 求证：$(0\leqslant x\leqslant 1)$.

(1) $\sum_{k=0}^{n}kC_n^k x^k(1-x)^{n-k}=nx$;

附件 D

107-11-15 (Thursday)

(3) $\lim\limits_{x\to 0}(\cos \pi x)^{\frac{1}{x^2}}$; (4) $\lim\limits_{x\to 0}\left(\frac{\operatorname{tg} x}{x}\right)^{\frac{1}{x^2}}$;

(5) $\lim\limits_{x\to +\infty}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}}$; (6) $\lim\limits_{x\to +\infty}\left(\frac{\pi}{2}-\operatorname{arc\,tg} x\right)^{\frac{1}{x}}$;

(7) $\lim\limits_{x\to 0}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$; (8) $\lim\limits_{x\to +\infty}\left(\operatorname{tg}\frac{\pi x}{2x+1}\right)^{\frac{1}{x}}$.

5.2.3 求下列极限(a、b 为实数):

(1) $\lim\limits_{x\to 0+}\left(\frac{1+x^a}{1+x^b}\right)^{\frac{1}{\ln x}}$; (2) $\lim\limits_{x\to +\infty}\left(\frac{1+x^a}{1+x^b}\right)^{\frac{1}{\ln x}}$.

5.2.4 设 $f(x)$ 二阶可导. 求证:

$$\lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}=f''(x).$$

5.2.5 设 $f(x)$ 在 $[a,+\infty)$ 上有界, $f'(x)$ 存在, 且 $\lim\limits_{x\to +\infty}f'(x)=b$. 求证: $b=0$.

5.2.6 设 $f(x)$ 在 $(a,+\infty)$ 上可导, 且 $\lim\limits_{x\to +\infty}[f(x)+f'(x)]=k$ (k 为有限或 $\pm\infty$). 求证: $\lim\limits_{x\to +\infty}f(x)=k$.

5.2.7 由拉格朗日中值定理,

$$\ln(1+x)-0=x\cdot\frac{1}{1+\theta x}\quad(0<\theta<1).$$

求证: $\lim\limits_{x\to 0}\theta=\frac{1}{2}$.

5.2.8 由拉格朗日中值定理,

$$e^x-1=xe^{\theta x}\quad(0<\theta<1).$$

求证: $\lim\limits_{x\to 0}\theta=\frac{1}{2}$.

5.2.9 (1) 求证: $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\frac{1}{3}$;

(2) 由拉格朗日中值定理,

附件 D

107-11-15 (Thursday)

$$\arcsin x - 0 = x \cdot \frac{1}{\sqrt{1-\theta^2 x^2}} \quad (0<\theta<1).$$

求证：$\lim\limits_{x\to 0}\theta = \frac{1}{\sqrt{3}}$.

皮亚诺余项的泰勒公式

5.3.1 写出下列函数在 $x=0$ 的带有皮亚诺余项的泰勒展式：

(1) e^{2x};　　(2) $\cos x^2$;

(3) $\ln(1-x)$;　　(4) $\frac{1}{(1+x)^2}$;

(5) $\frac{x^3+2x+1}{x-1}$;　　(6) $\sin^3 x$.

5.3.2 写出下列函数在 $x=0$ 的泰勒公式至所指的阶数：

(1) $1-x+x^2$　(x^3);　　(2) $e^x\cos x$　(x^4);

(3) $\frac{x}{\sin x}$　(x^4);　　(4) $\ln(\cos x+\sin x)$　(x^4);

(5) $\frac{x}{2x^2+x-1}$　(x^3);　　(6) $\frac{1+x+x^2}{1-x+x^2}$　(x^4);

(7) $\ln(1+x+x^2+x^3)$　(x^6);　　(8) $\ln\frac{1+x}{1-2x}$　(x^n).

5.3.3 在 $x=0$ 处将下列函数展开到 x^4.

(1) $\frac{x^2}{\sqrt{1-x+x^2}}$;　　(2) $\frac{1}{\sqrt{1-x^2+x^4}}$.

5.3.4 设 $f(x)$ 在 x_0 点 n 次可导，且

$$f(x)=\sum_{k=0}^{n}a_k(x-x_0)^k+o((x-x_0)^n).$$

求证：

$$f'(x)=\sum_{k=0}^{n-1}(k+1)a_{k+1}(x-x_0)^k+o((x-x_0)^{n-1}).$$

附件 D

20.4.6 对下列区域依两种不同顺序将二重积分 $\iint\limits_{\Omega} f(x,\ y)\,dx\,dy$ 化为累次积分:

(1) Ω 是以 $A_1(a_1, b_1)$, $A_2(a_2, b_2)$ 和 $A_3(a_3,\ b_3)$ $(a_1<a_2,\ b_1<b_2)$ 为顶点的三角形;

(2) Ω 是以 $A(0,0)$, $B(1,0)$, $C(-1,1)$ 为顶点的三角形;

(3) Ω 是以 $A(a,0)$, $B(2a,2a)$, $C(2a,4a)$ 和 $D(a,2a)$ 为顶点的四边形 $(a>0)$;

(4) Ω 是圆域 $(x-a)^2+(y-b)^2\leqslant R^2$;

(5) Ω 是圆域 $x^2+y^2\leqslant 2x$;

(6) Ω 是环域 $R_1^2\leqslant(x-a)^2+(y-b)^2\leqslant R_2^2\ (R_2>R_1)$;

(7) Ω 是由 $x^2+y^2=R^2$ 和 $x^2+y^2=Rx\ (R>0)$ 围成的区域;

(8) Ω 是由 $y=x^3$, $y=2x^3$, $y=1$ 和 $y=2$ 围成的区域;

(9) Ω 是由 $y=x^{\frac{1}{3}}$, $y=\left(\dfrac{x}{3}\right)^{\frac{1}{3}}$, $x=-1$ 和 $x=1$ 围成的区域.

20.4.7 在下列积分中改变积分的顺序:

(1) $\displaystyle\int_0^1 dy\int_0^y f(x,y)\,dx$;

(2) $\displaystyle\int_1^e dx\int_0^{\ln x} f(x,y)\,dy$;

(3) $\displaystyle\int_0^2 dy\int_{y^2}^{3-y} f(x,y)\,dx$;

(4) $\displaystyle\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy$;

(5) $\displaystyle\int_1^2 dx\int_{\sqrt{x}}^2 f(x,y)\,dy$;

(6) $\displaystyle\int_1^2 dx\int_{2-x}^{\sqrt{2x-a}} f(x,y)\,dy \quad (0<a<1)$;

(7) $\displaystyle\int_0^{\pi} dy\int_{\sin\frac{y}{2}}^{\sin y} f(x,y)\,dx$.

20.4.8 证明 Dirichlet 公式

$$\int_0^a dx\int_0^x f(x,y)\,dy=\int_0^a dy\int_y^a f(x,y)\,dx\,(a>0).$$

20.4.9 设一元函数 $g(x)$在[0,1]可积. 证明

$$\int_0^1 dx\int_x^1 g(t)\,dt=\int_0^1 tg(t)\,dt.$$

20.4.10 设 m 和 n 是正整数，且至少有一个是奇数. 证明

$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} x^m y^n\,dx\,dy=0.$$

20.4.11 设 m 和 n 是正整数，且都是偶数. 证明

$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} x^m y^n\,dx\,dy=4\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1,\\ x\geqslant 0,\ y\geqslant 0}} x^m y^n\,dx\,dy.$$

20.4.12 计算下列二重积分:

(1) Ω 是由 $y^2=2px(p>0)$,$x=\frac{p}{2}$围成的区域，计算

$$\iint\limits_{\Omega} x^m y^k\,dx\,dy\,(m,k>0);$$

(2) Ω 由 $y=0$, $y=\sin(x^2)$, $x=0$ 和 $x=\sqrt{\pi}$ 围成，计算

$$\iint\limits_{\Omega} x\,dx\,dy;$$

(3) $\Omega=\{(x,y)\,|\,0\leqslant x\leqslant y^2,0\leqslant y\leqslant 2+x,x\leqslant 2\}$，计算

$$\iint\limits_{\Omega} x^2y^2\,dx\,dy;$$

(4) Ω由 $y=\sqrt{1-x^2}$, $y=0$ 围成，计算$\iint\limits_{\Omega}(x^2+3xy^2)\,dx\,dy$;

(5) Ω由 $y=e^x$, $y=1$, $x=0$ 及 $x=1$ 围成，计算

$$\iint\limits_{\Omega}(x+y)\,dx\,dy;$$

These calculations give the surface integral

$$\iint_S \sqrt{1 - x^2 - y^2}\, d\sigma = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= 2 \int_0^{2\pi} \int_0^{\pi/2} \sin u \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{w}\, dw\, dv \qquad \begin{aligned} &w = 1 + \sin^2 u, \\ &dw = 2 \sin u \cos u\, du \\ &\text{When } u = 0, w = 1. \\ &\text{When } u = \pi/2, w = 2. \end{aligned}$$

$$= 2\pi \cdot \frac{2}{3} w^{3/2} \Big]_1^2 = \frac{4\pi}{3}\left(2\sqrt{2} - 1\right).$$

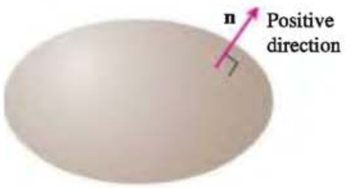


FIGURE 16.49 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

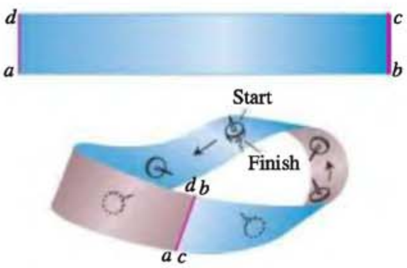


FIGURE 16.50 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\mathbf{n}$ of unit normal vectors on S that varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.49).

The Möbius band in Figure 16.50 is not orientable. No matter where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

Surface Integral for Flux

Suppose that $\mathbf{F}$ is a continuous vector field defined over an oriented surface S and that $\mathbf{n}$ is the chosen unit normal field on the surface. We call the integral of $\mathbf{F} \cdot \mathbf{n}$ over S the flux of $\mathbf{F}$ across S in the positive direction. Thus, the flux is the integral over S of the scalar component of $\mathbf{F}$ in the direction of $\mathbf{n}$.

DEFINITION The **flux** of a three-dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \tag{5}$$

The definition is analogous to the flux of a two-dimensional field $\mathbf{F}$ across a plane curve C. In the plane (Section 16.2), the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

the integral of the scalar component of $\mathbf{F}$ normal to the curve.

If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, the flux of $\mathbf{F}$ across S is the net rate at which fluid is crossing S in the chosen positive direction. We discuss such flows in more detail in Section 16.7.

附件 D

This page will be the front page of Quiz 12 on May 23

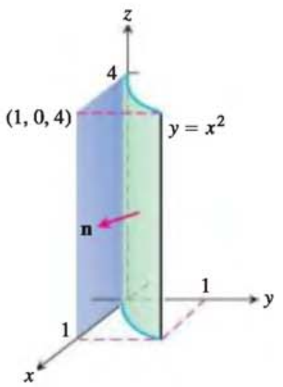


FIGURE 16.51 Finding the flux through the surface of a parabolic cylinder (Example 4).

EXAMPLE 4 Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2, 0 \le x \le 1, 0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in Figure 16.51.

Solution On the surface we have $x = x, y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, 0 \le x \le 1, 0 \le z \le 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The unit normal vectors pointing outward from the surface as indicated in Figure 16.51 are

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Thus,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{4x^2 + 1}}((x^2z)(2x) + (x)(-1) + (-z^2)(0)) \\ &= \frac{2x^3z - x}{\sqrt{4x^2 + 1}}. \end{aligned}$$

The flux of $\mathbf{F}$ outward through the surface is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} |\mathbf{r}_x \times \mathbf{r}_z|\, dx\, dz \\ &= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} \sqrt{4x^2 + 1}\, dx\, dz \\ &= \int_0^4 \int_0^1 (2x^3z - x)\, dx\, dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2\right]_{x=0}^{x=1} dz \\ &= \int_0^4 \frac{1}{2}(z - 1)\, dz = \frac{1}{4}(z - 1)^2\Big]_0^4 \\ &= \frac{1}{4}(9) - \frac{1}{4}(1) = 2. \end{aligned}$$

If S is part of a level surface $g(x, y, z) = c$, then $\mathbf{n}$ may be taken to be one of the two fields

$$\mathbf{n} = \pm\frac{\nabla g}{|\nabla g|}, \tag{6}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \\ &= \iint_R \left(\mathbf{F} \cdot \frac{\pm\nabla g}{|\nabla g|}\right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|}\, dA \qquad \text{Eqs. (6) and (3)} \\ &= \iint_R \mathbf{F} \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA. \end{aligned} \tag{7}$$

附件 D

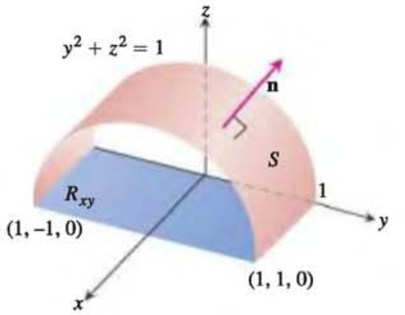


FIGURE 16.52 Calculating the flux of a vector field outward through the surface S. The area of the shadow region R_{xy} is 2 (Example 5).

EXAMPLE 5 Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution The outward normal field on S (Figure 16.52) may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\mathbf{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{1}} = y\mathbf{j} + z\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$, we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{2}{|2z|}\, dA = \frac{1}{z}\, dA.$$

We can drop the absolute value bars because $z \geq 0$ on S.

The value of $\mathbf{F} \cdot \mathbf{n}$ on the surface is

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) \\ &= y^2 z + z^3 = z(y^2 + z^2) \\ &= z. \qquad y^2 + z^2 = 1 \text{ on } S \end{aligned}$$

The surface projects onto the shadow region R_{xy}, which is the rectangle in the xy-plane shown in Figure 16.52. Therefore, the flux of $\mathbf{F}$ outward through S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S (z)\left(\frac{1}{z}\, dA\right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2.$$

Moments and Masses of Thin Shells

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated with the formulas in Table 16.3. The derivations are similar to those in Section 6.6. The formulas are like those for line integrals in Table 16.1, Section 16.1.

TABLE 16.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta\, d\sigma$ $\quad \delta = \delta(x, y, z) =$ density at (x, y, z) as mass per unit area

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\,\delta\, d\sigma, \qquad M_{xz} = \iint_S y\,\delta\, d\sigma, \qquad M_{xy} = \iint_S z\,\delta\, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2 + z^2)\,\delta\, d\sigma, \quad I_y = \iint_S (x^2 + z^2)\,\delta\, d\sigma, \quad I_z = \iint_S (x^2 + y^2)\,\delta\, d\sigma,$$

$$I_L = \iint_S r^2\delta\, d\sigma$$ $\quad r(x, y, z) =$ distance from point (x, y, z) to line L

附件 D

Quiz 1 (2019-02-21) Name(Print) ____________________

1. State and prove the Bolzano-Weierstrass theorem in $\mathbb{R}$.

2. State the triangle inequality and the Cauchy inequality in $\mathbb{R}^m$ and prove that they are equivalent.

附件 D

Quiz 2 (2019-03-07) Name(Print) ____________________

1. Def of limit in $\mathbb{R}^m$, with set-up.

---

2. Def of continuity in $\mathbb{R}^m$, with set-up.

---

3. State and prove the max-min theorem in $\mathbb{R}^m$.

4. A complete statement and proof that any two norms on $\mathbb{R}^m$ are equivalent.

Statement:

Proof. (Including the reduction to the Euclidean norm and the continuity of a general norm.)

1. Def of a norm and a metric (including setup and axioms).
   - **Norm setup:**

   - **Norm Axioms:**

   - **Metric setup:**

   - **Metric Axioms:**

Disclaimer: For the rest of the quiz, we live in a metric space.

---

2. Show that the complement of an open set is a closed set and the complement of a closed set is open.

3. State and prove Theorem 3 in p. 175 on $\bar{E}$.

---

4. Show that $\partial E = \partial(E^c)$.

---

5. Open sets: Finite intersections and countable unions are still open.
6. Closed sets: Finite unions and countable intersections are still closed.

附件 D

1. Let (X, d) be a metric space, and let $E \subset X$ be a subset. Show that E^{int} is open.

---

2. Show that the union of open sets is open, and the finite intersection of open sets is still open. Is the conclusion true for infinite intersection? Then state the corresponding statement for closed sets.

3. State and prove THE important theorem on compactness in $\mathbb{R}^m$. That is, TFAE:

(1)

(2)

(3)

$(1) \to (2)$

$(2) \to (3)$

$(3) \to (1)$

附件 **D**

Quiz 5 (2019-03-28) Name(Print) ____________________

1. Let (X, d) be a metric space, let $K \subset X$ be compact, and let $f \in C(K)$.

   (1) f is bounded.

---

   (2) f is uniformly continuous.

---

2. Let $H = \{c = (c_1, c_2, \cdots) \in \mathbb{R}^\infty : \sum_{k=1}^{\infty} c_k^2 < \infty\}$ with metric $d(c, \tilde{c}) = \sqrt{\sum_{k=1}^{\infty}(c_k - \tilde{c}_k)^2}$. Show that (a) H is complete; (b) its closed unit ball is not compact.

附件 **D**

3. State and prove the theorem which says that continuous partial derivatives imply differentiability.

   Statement:

   Proof.

附件 **D**

1. State and prove the Clairaut's theorem.

The technical version:

The sleek version:

Proof.

2. State and prove the implicit function theorem. Use an extra sheet since you need it

Statement:

Proof.

- At least one pic is required.
- It is strongly suggested that you enumerate your steps.

附件 **D**

Quiz 7 (2019-04-18) Name(Print) ____________________________________

1. (a) State and prove the lemma for positive definite quadratic form (p. 273).

(b) State the conditions for

$$f(a+h) - f(a) = \frac{1}{2}\sum A_{ij}h_i h_j + o(||h||^2)$$

and prove it (p. 270).

(c) State and prove the theorem on strict local min (Theorem 2, p. 274).

2. State the full version of the Lagrange multiplier theorem (Theorem 3, p. 277).

Then prove the case $p = 1$.

20.4.6 对下列区域依两种不同顺序将二重积分$\iint\limits_{\Omega} f(x, y)\,dx\,dy$化为累次积分：

(1) Ω 是以 $A_1(a_1, b_1)$, $A_2(a_2, b_2)$ 和 $A_3(a_3, b_3)$ $(a_1<a_2,\ b_1<b_2)$ 为顶点的三角形；

(2) Ω 是以 $A(0,0)$, $B(1,0)$, $C(-1,1)$ 为顶点的三角形；

(3) Ω 是以 $A(a,0)$, $B(2a,2a)$, $C(2a,4a)$ 和 $D(a,2a)$ 为顶点的四边形$(a>0)$；

(4) Ω 是圆域$(x-a)^2+(y-b)^2\leqslant R^2$；

(5) Ω 是圆域 $x^2+y^2\leqslant 2x$；

(6) Ω 是环域 $R_1^2\leqslant (x-a)^2+(y-b)^2\leqslant R_2^2\ (R_2>R_1)$；

(7) Ω 是由 $x^2+y^2=R^2$ 和 $x^2+y^2=Rx$ $(R>0)$ 围成的区域；

(8) Ω 是由 $y=x^3$, $y=2x^3$, $y=1$ 和 $y=2$ 围成的区域；

(9) Ω 是由 $y=x^{\frac{1}{3}}$, $y=\left(\frac{x}{3}\right)^{\frac{1}{3}}$, $x=-1$ 和 $x=1$ 围成的区 域.

20.4.7 在下列积分中改变积分的顺序：

(1) $\int_0^1 dy\int_0^y f(x,y)\,dx$；

(2) $\int_1^e dx\int_0^{\ln x} f(x,y)\,dy$；

(3) $\int_0^2 dy\int_{y^2}^{3y} f(x,y)\,dx$；

(4) $\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy$；

(5) $\int_1^2 dx\int_{\sqrt{x}}^2 f(x,y)\,dy$；

(6) $\int_1^2 dx\int_{2-x}^{\sqrt{2x-a}} f(x,y)\,dy\quad (0<a<1)$；

(7) $\int_0^{\pi} dy\int_{\sin\frac{y}{2}}^{\sin y} f(x,y)\,dx$.

20.4.8 证明 Dirichlet 公式

$$\int_0^a dx\int_0^x f(x,y)\,dy=\int_0^a dy\int_y^a f(x,y)\,dx\,(a>0).$$

20.4.9 设一元函数 $g(x)$在[0,1]可积. 证明

$$\int_0^1 dx\int_x^1 g(t)\,dt=\int_0^1 tg(t)\,dt.$$

20.4.10 设 m 和 n 是正整数,且至少有一个是奇数. 证明

$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} x^m y^n\,dx\,dy=0.$$

20.4.11 设 m 和 n 是正整数,且都是偶数. 证明

$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} x^m y^n\,dx\,dy=4\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1,\\ x\geqslant 0,\ y\geqslant 0}} x^m y^n\,dx\,dy.$$

20.4.12 计算下列二重积分:

(1) Ω 是由 $y^2=2px(p>0)$,$x=\frac{p}{2}$围成的区域,计算

$$\iint\limits_{\Omega} x^m y^k\,dx\,dy\,(m,k>0);$$

(2) Ω 由 $y=0$, $y=\sin(x^2)$, $x=0$ 和 $x=\sqrt{\pi}$ 围成, 计算

$$\iint\limits_{\Omega} x\,dx\,dy;$$

(3) $\Omega=\{(x,y)\,|\,0\leqslant x\leqslant y^2, 0\leqslant y\leqslant 2+x, x\leqslant 2\}$, 计算

$$\iint\limits_{\Omega} x^2y^2\,dx\,dy;$$

(4) Ω 由 $y=\sqrt{1-x^2}$, $y=0$ 围成, 计算$\iint\limits_{\Omega}(x^2+3xy^2)\,dx\,dy$;

(5) Ω 由 $y=e^x$, $y=1$, $x=0$ 及 $x=1$ 围成,计算

$$\iint\limits_{\Omega}(x+y)\,dx\,dy;$$

A

第二型曲线积分

21.3.1 计算第二型曲线积分 $\int_{\widehat{AB}} y\,dx + x\,dy$，$A(1,1)$，$B(2,4)$，方向从 A 到 B：

(1) $\widehat{AB}$ 为连结 A 与 B 点的直线段 $\overline{AB}$；

(2) $\widehat{AB}$ 为连结 A 与 B 点的抛物线段 $y=x^2 \quad (1\leqslant x\leqslant 2)$；

(3) $\widehat{AB}$ 为折线段：$\overline{AC}+\overline{CB}$，其中 $C(2,1)$，方向从 A 到 C，再从 C 到 B.

21.3.2 计算 $\int_{\widehat{AB}} y\,dx - x\,dy$，$A$ 点，B 点以及弧 $\widehat{AB}$ 同 21.3.1.

21.3.3 求力场 $\boldsymbol{F}$ 对运动的单位质点所作的功，此质点沿曲线 C 从 A 点运动到 B 点：

(1) $\boldsymbol{F}=(x-2xy^2)\boldsymbol{i}+(y-2x^2y)\boldsymbol{j}$，$C$ 为平面曲线 $y=x^2$，$A(0,0)$，$B(1,1)$；

(2) $\boldsymbol{F}=(x+y)\boldsymbol{i}+xy\boldsymbol{j}$，$C$ 为平面曲线 $y=1-|1-x|$，$A(0,0)$，$B(2,0)$；

(3) $\boldsymbol{F}=(x-y)\boldsymbol{i}+(y-z)\boldsymbol{j}+(z-x)\boldsymbol{k}$，$C$ 的 $C^{(1)}$ 参数式为 $\boldsymbol{r}(t)=t\boldsymbol{i}+t^2\boldsymbol{j}+t^3\boldsymbol{k}$，$A(0,0,0)$，$B(1,1,1)$；

(4) $\boldsymbol{F}=y^2\boldsymbol{i}+z^2\boldsymbol{j}+x^2\boldsymbol{k}$，$C$ 的参数式为 $x=\alpha\cos t$，$y=\beta\sin t$，$z=\gamma t$（α,β,γ 为正数），$A(\alpha,0,0)$，$B(\alpha,0,2\pi\gamma)$.

21.3.4 计算下列曲线积分

$$\int_C (y^2-z^2)\,dx+(z^2-x^2)\,dy+(x^2-y^2)\,dz.$$

(1) C 为球面三角形 $x^2+y^2+z^2=1$，$x\geqslant 0$，$y\geqslant 0$，$z\geqslant 0$ 的边界线，从球的外侧看去，C 的方向为逆时钟方向；

(2) C 是球面 $x^2+y^2+z^2=a^2$ 和柱面 $x^2+y^2=ax\,(a>0)$ 的交线位于 xy 平面上方的部分，从 x 轴上 $(b,0,0)\,(b>a)$ 点看去，C 的方向是顺时钟方向.

Quiz 10 on 05-09 will be three randomly chosen problems from this page.

B

21.3.5 求闭曲线 C 上的第二型曲线积分

$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}.$$

(1) C 为圆 $x^2+y^2=a^2$，逆时钟方向；

(2) C 为椭圆 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$，顺时钟方向；

(3) C 是以(0, 0)为中心而边长为 a 的正方形，顺时钟方向；

(4) C 是以(−1, −1), (1, −1), (0, 1)为顶点的三角形，顺时钟方向.

21.3.6 求闭曲线 C 上的第二型曲线积分

$$\oint_C \frac{x\,dx + y\,dy}{x^2+y^2},$$

C 及 C 的方向同 **21.3.5.**

21.3.7 设 P, Q, R 在 L 上连续，L 为光滑弧段，弧长为 l. 证明

$$\left|\int_L p\,dx + Q\,dy + R\,dz\right| \leqslant Ml,$$

其中
$$M = \max_{(x,y,z)\in L}\{\sqrt{P^2+Q^2+R^2}\}.$$

21.3.8. 设 L 是圆周 $x^2+y^2=R^2$，计算

$$\int_L \frac{x\,dy - y\,dx}{Ax^2-12Bxy+Cy^2}$$

(A, C 为正数，$AC-B^2>0$)，L 取正向.

Quiz 10 (May 09) Name:

21.3.3 求力场 $\boldsymbol{F}$ 对运动的单位质点所作的功，此质点沿曲线 C 从 A 点运动到 B 点：

(1) $\boldsymbol{F}=(x-2xy^2)\boldsymbol{i}+(y-2x^2y)\boldsymbol{j}$，$C$ 为平面曲线 $y=x^2$，$A(0,0)$，$B(1,1)$；

(2) $\boldsymbol{F}=(x+y)\boldsymbol{i}+xy\boldsymbol{j}$，$C$ 为平面曲线 $y=1-|1-x|$，$A(0,0)$，$B(2,0)$；

附件 D

21.3.4 计算下列曲线积分

$$\int_C (y^2-z^2)\,dx+(z^2-x^2)\,dy+(x^2-y^2)\,dz.$$

(1) C 为球面三角形 $x^2+y^2+z^2=1, x\geqslant 0, y\geqslant 0, z\geqslant 0$ 的边界线，从球的外侧看去，C 的方向为逆时针方向；

(2) C 是球面 $x^2+y^2+z^2=a^2$ 和柱面 $x^2+y^2=ax(a>0)$ 的交线位于 xy 平面上方的部分，从 x 轴上 $(b,0,0)(b>a)$ 点看去，C 的方向是顺时针方向.

附件 D

Quiz 11 (May 16) Name:

$$\oint_{\partial D^+} (x^2+y^3)\,dx-(x^3-y^2)\,dy,\ D:\ \ x^2+y^2\leqslant 1;$$

$$\oint_{\partial D^+} (e^y \sin x\,dx+e^{-x}\sin y\,dy),\ D:\ \ a\leqslant x\leqslant b,\ c\leqslant y\leqslant d;$$

$\oint_L f(x^2+y^2)(x\,dx+y\,dy)=$

21.4.6 设 C 为光滑的闭曲线，$\boldsymbol{a}$ 为任一固定的单位向量. 证明 $\oint_C \cos\langle \boldsymbol{a}, \boldsymbol{n}\rangle ds=0$，其中 $\boldsymbol{n}$ 为 C 的外法线方向.

Quiz 12 on May 23 Name:

EXAMPLE 4 Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2, 0 \le x \le 1, 0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in Figure 16.51.

Solution On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, 0 \le x \le 1, 0 \le z \le 4$. The cross product of tangent vectors is

The unit normal vectors pointing outward from the surface as indicated in Figure 16.51 are

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} =$$

Thus,

$$\mathbf{F} \cdot \mathbf{n} =$$

The flux of $\mathbf{F}$ outward through the surface is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma =$$

例 7 试计算 $L=\iint\limits_{S}x^{2}\mathrm{d}y\mathrm{d}z+y^{2}\mathrm{d}z\mathrm{d}x+z^{2}\mathrm{d}x\mathrm{d}y$,

这里 S 是球面 $x^{2}+y^{2}+z^{2}=a^{2}$ 的外侧.

解 类似于例 4 中的做法，我们求得

例 8 试计算

$$M=\iint\limits_{\Gamma}f(x)\mathrm{d}y\mathrm{d}z+g(y)\mathrm{d}z\mathrm{d}x+h(z)\mathrm{d}x\mathrm{d}y,$$

这里 Γ 是以下长方体的外表面：

$$|x|\leqslant a,\quad |y|\leqslant b,\quad |z|\leqslant c.$$

解 仿照例 5 中的做法，我们求得

附件 D

Quiz 13 (2019-05-30) Name(Print) ____________________

1. Q: What is the Stokes theorem?

---

2. Give the complete statement of an example of Stokes' theorem. (Hint: Green)

---

3. Give the complete statement of another example of Stokes' theorem. (Hint: Gauss)

---

4. Explain that FTC is also an example of Stokes!

附件 D

5. Find the surface area: $x = r\cos\varphi, y = r\sin\varphi, z = \varphi$, where $0 < r < a, 0 < \varphi < 2\pi$!
Roughly explain the shape of the surface.

---

6. Find the flux of the vector field $F = y\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ through the boundary of $\Omega = \{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$, with interior normal.

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7. Find the flux of the vector field $F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the surface $S = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z = 1\}$, with downward normal.

Quiz 14 (June 6)　　Name:

11.2.7 利用泰勒级数估计无穷小量 a_n 的阶，从而判别下列级数的收敛性：

(1) $\sum_{n=1}^{+\infty} 2^n \sin \frac{\pi}{3^n}$;

(2) $\sum_{n=1}^{+\infty} \frac{1}{\ln(n+1)} \sin \frac{1}{n}$;

(3) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3+1}}$;

(4) $\sum_{n=1}^{+\infty} (\sqrt{n+1}-\sqrt{n})^p \ln \frac{n-1}{n+1}$;

(5) $\sum_{n=3}^{+\infty} \ln^p \cos \frac{\pi}{n}$;

(6) $\sum_{n=1}^{+\infty} (\sqrt{n+a}-\sqrt[4]{n^2+n+b})$;

(7) $\sum_{n=1}^{+\infty} \left[e-\left(1+\frac{1}{n}\right)^n \right]^p$

附件 D

11.2.8 若级数$\sum_{n=1}^{+\infty}a_n$ $(a_n>0)$发散，求证：级数$\sum_{n=1}^{+\infty}\frac{a_n}{S_n}$也发散。

11.2.9 若正项级数$\sum_{n=1}^{+\infty}a_n$收敛，$a_{n+1}\leqslant a_n(n=1,2,\cdots)$，求证：$\lim\limits_{n\to+\infty}n\cdot a_n=0$。

11.2.11 设$0<P_1<P_2<\cdots<P_n<\cdots$，求证：$\sum_{n=1}^{+\infty}\frac{1}{P_n}$收敛的充要条件为级数$\sum_{n=1}^{+\infty}\frac{n}{P_1+P_2+\cdots+P_n}$收敛.

附件 D

1. State three definitions of $\limsup x_n$ and show that they are all equivalent.

- Def I:

- Def II:

- Def III:

Proof.

12.3.2 研究下列级数在什么区间上一致收敛

(1) $\sum_{n=1}^{+\infty}\frac{1}{1+n^2x^2}$;

(3) $\sum_{n=1}^{+\infty}\frac{\cos nx}{n^2}$;

(5) $\sum_{n=1}^{+\infty}\frac{x^2}{(1+x^2)^n}$.

12.3.3 设 $f_n(x)=\frac{x^2}{x^2+(1-nx)^2}, 0\leqslant x\leqslant 1$. 求证：它的任何子序列都不一致收敛.

附件 D